Section 10.3: Series with Positive Terms

Throughout this lecture, we will sum series with non-negative terms only.

\[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots \]

\[ a_n \geq 0 \text{ for all } n \]

**Theorem #1**

If \( a_n > 0 \) for all \( n \) with \( S_n = \sum_{n=1}^{N} a_n \) and \( S = \lim_{N \to \infty} S_n \), then exactly one of the following is true:

1. \( S_n \) is bounded above, hence \( S \) converges.
2. \( S_n \) is not bounded above, hence \( S = \infty \).

**Proof:** Since \( a_n > 0 \), \( \{S_n\} \) is increasing. Now use Theorem #2 of Section 10.1.
Careful! Not true for general series.

\[ \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \ldots \]

Partial sums are bounded, but series diverges (geometric with \( r = -1 \)).

**Theorem #2 (Integral Test)**

Let \( a_n = f(n) \) and suppose \( f(x) \) is:

1. **non-negative** \( f(x) \geq 0 \)
2. **decreasing** \( x < y \implies f(x) > f(y) \)
3. **continuous** \( f(x) \) is cont.

Then \( \sum_{n=1}^{\infty} a_n \) converges \( \iff \int_{1}^{\infty} f(x) \, dx \) converges.

**Proof:**

Consider the left and right Riemann sums for \( \int_{1}^{N} f(x) \, dx \) by partitioning \([1, N]\) into \( N \) equal subintervals.
Since \( \text{f}(x) \) is decreasing we have:

\[
R_N \leq \int_1^N \text{f}(x) \, dx \leq L_N
\]
Hence we have, for all \( N \):

\[
S_N - a_1 \leq \int_1^N f(x) \, dx \leq S_{N-1}
\]

- Suppose \( \int_1^\infty f(x) \, dx \) converges. Then \( S_N - a_1 \leq \int_1^N f(x) \, dx \leq \int_1^\infty f(x) \, dx \). So \( S_N \leq a_1 + \int_1^\infty f(x) \, dx \) for all \( N \). Hence \( \{S_N\} \) is bounded above. By Theorem \#1, \( \lim_{N \to \infty} S_N = \sum_{n=1}^{\infty} a_n \) converges since \( a_n \geq 0 \) for all \( n \).

- Suppose \( \int_1^\infty f(x) \, dx \) diverges. Then the sequence \( b_n = \int_1^N f(x) \, dx \) is increasing (since \( f(x) \geq 0 \)) and grows without bound. Since \( \int_1^N f(x) \, dx \leq S_{N-1} \) for all \( N \), \( \{S_{N-1}\} \) is not bounded above. Hence by Theorem \#1, \( \lim_{N \to \infty} S_N = \infty \), so \( \sum_{n=1}^{\infty} a_n \) diverges.
Note: If $f(x)$ satisfies the hypotheses of the Integral Test, then we always have

\[ S_N - a_1 \leq \int_1^N f(x) \, dx \leq S_{N-1} \]

... or, equivalently...

\[ f(N) + \int_1^N f(x) \, dx \leq S_N \leq f(1) + \int_1^N f(x) \, dx \]

This is true even if the series and integral diverge. This inequality can be used to estimate convergent sums or to estimate how fast a divergent series grows.

Ex: Let \( S_N = \sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \). Then we have from the above inequality:

\[ \frac{1}{N} + \int_1^N \frac{1}{x} \, dx \leq S_N \leq 1 + \int_1^N \frac{1}{x} \, dx \]

\[ \frac{1}{N} + \ln(N) \leq S_N \leq 1 + \ln(N) \]

\( S_N \) grows at the same rate as \( \ln(N) \).
Ex. 1

Determine convergence of

\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \]

(Harmonic series.)

Solution:

Let \( f(x) = \frac{1}{x} \). Use Integral Test.

Observe the following:

1. \( f(x) \geq 0 \) (for \( x > 0 \))

2. \( f(x) \) is decreasing

\( f'(x) = -\frac{1}{x^2} < 0 \) (for all \( x \))

So \( f \) is decreasing on \( (0, \infty) \)
(3) $f(x)$ is cont. since it is diff.
Now we have
\[
\int_1^\infty \frac{1}{x} \, dx = \lim_{R \to \infty} \ln(x) \bigg|_1^R = \lim_{R \to \infty} \ln(R) = \infty
\]
Since integral diverges, series diverges.

**Theorem #3 (p-test)**
The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff p > 1$.

**Proof:**

$p > 0$: Use Integral Test and p-test for Integrals from Section 7.7 notes.

$p \leq 0$: Use Nth-term divergence test. $\square$

**Ex. 2**
Determine whether series converges.
\[
\sum_{n=1}^{\infty} n e^{-n^2}
\]

**Solution:**
Use Integral Test. Let $f(x) = xe^{-x^2}$. 

Observe the following:

1. \( f(x) > 0 \) (for \( x > 0 \))

2. \( f \) is decreasing

\[
\begin{align*}
    f''(x) &= xe^{-x^2}(-2x) + e^{-x^2} \\
    &= xe^{-x^2}(1 - 2x^2) \\
    f'(x) &= e^{-x^2}(1 - 2x^2)
\end{align*}
\]

Note that \( f'(x) < 0 \) for \( 1 - 2x^2 < 0 \), or \( x > \frac{1}{\sqrt{2}} \). So \( f \) is decreasing on \((1, \infty)\).

3. \( f \) is cont. since it is diff.

Now we have

\[
\int_1^\infty xe^{-x^2} \, dx = \lim_{R \to \infty} \left( -\frac{1}{2} e^{-x^2} \right)_{1}^{R}
\]

\[
= \frac{1}{2e} - \lim_{R \to \infty} \frac{1}{2eR^2} = \frac{1}{2e}
\]

Integral converges, so the series converges.
Reminder: The value of $\sum_{n=1}^{\infty} n e^{-n^2}$ is not $\frac{1}{2e}$. All we know from the earlier approximation is that 

$$\int_{1}^{\infty} xe^{-x^2} \, dx \leq \sum_{n=1}^{\infty} ne^{-n^2} \leq f(1) + \int_{1}^{\infty} xe^{-x^2} \, dx$$

$$\Rightarrow \quad \frac{1}{2e} \leq \sum_{n=1}^{\infty} ne^{-n^2} \leq \frac{1}{e} + \frac{1}{2e}$$

Theorem #4: (Direct Comparison Test / DCT)

Suppose $0 \leq a_n \leq b_n$ for all $n$.

(a) If $\sum b_n$ converges, then $\sum a_n$ converges.

(b) If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Ex. 3

Determine whether series converges:
\[
\frac{1}{n \cdot 4^n}
\]

**Solution:**

Observe the following: (for all \( n \geq 1 \))

\[
\frac{1}{n \cdot 4^n} \leq \frac{1}{n} \cdot \frac{1}{4^n}
\]

\[
\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges by p-test} \quad (p = 1 \leq 1)
\]

\[
\frac{1}{n \cdot 4^n} \leq \frac{1}{4^n}
\]

\[
\sum_{n=1}^{\infty} \frac{1}{4^n} \text{ converges by geometric series test with } r = \frac{1}{4} \text{ and } |r| < 1.
\]

(The first inequality is useless. "Smaller than divergent" means nothing.)

Observe that \( \frac{1}{n \cdot 4^n} \leq \frac{1}{4^n} \) and \( \sum_{n=1}^{\infty} \frac{1}{4^n} \) converges by Geometric Series Test, with \( r = \frac{1}{4} \). So by DCT, \( \sum_{n=1}^{\infty} \frac{1}{n \cdot 4^n} \) converges.
Determine whether series converges.

\[ \sum_{n=4}^{\infty} \frac{1}{(n^2 - 3)^{1/3}} \]

Solution:

Observe that

\[ 0 \leq n^2 - 3 \leq n^2 \]

for \( n \geq 4 \)

\[ 0 \leq \frac{1}{n^{2/3}} \leq \frac{1}{(n^2 - 3)^{1/3}} \]

The series \( \sum_{n=4}^{\infty} \frac{1}{n^{2/3}} \) diverges by \( p \)-test \( (p = \frac{2}{3} \leq 1) \). So by DCT,

\[ \sum_{n=4}^{\infty} \frac{1}{(n^2 - 3)^{1/3}} \] diverges.

**Theorem #5** *(Limit Comparison Test / LCT)*

Suppose \( a_n \geq 0 \) and \( b_n > 0 \) for all \( n \). Let
\[ L = \lim_{n \to \infty} \frac{a_n}{b_n} \]

1. If 0 < L < \infty, then \( \Sigma a_n \) and \( \Sigma b_n \) both converge or both diverge.

2. If L = 0 and \( \Sigma b_n \) converges, then \( \Sigma a_n \) converges also.

3. If L = \infty and \( \Sigma b_n \) diverges, then \( \Sigma a_n \) diverges also.

**Ex. 5**

Determine convergence of

\[ \sum_{n=4}^{\infty} \frac{1}{(n^2 + 3)^{1/2}} \]

**Solution:**

DCT is difficult to use since

\[ 0 \leq \frac{1}{(n^2 + 3)^{1/2}} \leq \frac{1}{n^{2/3}} \]

Use LCT instead. Observe that:
\[
L = \lim_{n \to \infty} \frac{1}{(n^2 + 3)^{1/3}}
\]

Let's analyze the series you are testing goes in numerator. Series you already know goes in denominator.

\[
= \lim_{n \to \infty} \frac{n^{2/3}}{(n^2 + 3)^{1/3}} = \lim_{n \to \infty} \left( \frac{n^2}{n^2 + 3} \right)^{1/3}
\]

\[
= \lim_{n \to \infty} \left( \frac{1}{1 + 3/n^2} \right)^{1/3} = \left( \frac{1}{1 + 0} \right)^{1/3} = 1
\]

The series \( \sum_{n=4}^{\infty} \frac{1}{n^{2/3}} \) diverges by \( p \)-test \( (p = \frac{2}{3} \leq 1) \). So, since \( 0 < L < \infty \), by LCT, \( \sum_{n=4}^{\infty} \frac{1}{n^2 + 3}^{1/3} \) diverges.

**Ex. 6**

Determine convergence of

\[
\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n^9 + n + 1)^{1/4}}
\]
Solution:

If \( a_n \) is an algebraic function of \( n \), use LCT combined with \( \rho \)-test.

If \( n \) is very large,

\[
\frac{\sqrt{n}}{(n^q + n + 1)^{1/4}} \sim \frac{n^{1/2}}{(n^q)^{1/4}} \sim \frac{1}{n^{7/4}}
\]

Now use LCT.

\[
L = \lim_{n \to \infty} \frac{\sqrt{n}}{(n^q + n + 1)^{1/4}} = \frac{1}{n^{7/4}}
\]

\[
= \lim_{n \to \infty} \frac{n^{q/4}}{(n^q + n + 1)^{1/4}} = \lim_{n \to \infty} \left( \frac{n^q}{n^q + n + 1} \right)^{1/4}
\]

\[
= \lim_{n \to \infty} \left( \frac{1}{1 + \frac{1}{n^q} + \frac{1}{n^q}} \right)^{1/4} = 1
\]
The series \( \sum_{n=1}^{\infty} \frac{1}{n^{7/4}} \) converges by p-test \((p = 7/4 > 1)\). Since \(0 < L < \infty\), by LCT, \( \sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n^q + n+1)^{1/4}} \) converges.

Ex. 7

Determine convergence of each series:

(a) \( \sum_{n=2}^{\infty} \frac{\ln(n)}{n} \)  
(b) \( \sum_{n=2}^{\infty} \frac{1}{n^2 \ln(n)} \)  
(c) \( \sum_{n=2}^{\infty} \frac{\ln(n)}{n^2} \)  
(d) \( \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \)

Solution:

(a) Use DCT.  
\[ \ln(n) > 1 \]  (for \( n > 3 \))  
\[ \frac{\ln(n)}{n} > \frac{1}{n} \]
\[ \sum \frac{1}{n} \text{ diverges by p-test (} p=1 \leq 1 \)
\[ \sum \frac{\ln(n)}{n} \text{ diverges by DCT} \]

(b) Use DCT.

\[ \ln(n) > 1 \quad \text{(for } n \geq 3) \]

\[ \frac{1}{\ln(n)} < 1 \]

\[ \frac{1}{n^2 \ln(n)} < \frac{1}{n^2} \]

\[ \sum \frac{1}{n^2} \text{ converges by p-test } (p = 2 > 1) \]

\[ \sum \frac{1}{n^2 \ln(n)} \text{ converges by DCT.} \]

(c) Note that

\[ \frac{\ln(n)}{n^2} > \frac{1}{n^2} \]

This inequality is useless since \( \sum \frac{1}{n^2} \)

converges. What about LCT?

\[ L = \lim_{n \to \infty} \frac{\ln(n) / n^2}{1/n^2} = \lim_{n \to \infty} \frac{\ln(n)}{n} = \infty \]
This is also useless; $\sum \frac{1}{n^2}$ converges but $L=\infty$ is good only for testing divergence!

**General hierarchy:**
For any fixed $a > 0$, we have

$$\ln(n) < n^a$$

for large enough $n$.

So we can use DCT or LCT, but not with $\frac{1}{n^2}$. For large enough $n$,

$$\frac{\ln(n)}{n^2} < \frac{n^a}{n^2} = \frac{1}{n^{2-a}}$$

So just choose $a$ so that $a > 0$ and $2-a > 1$ (to get convergence). That is, choose $a$ so that $0 < a < 1$. What does this mean? Use DCT or LCT, but compare to $\sum \frac{1}{n^p}$ where $1 < p < 2$. 
\[ L = \lim_{n \to \infty} \frac{\ln(n)/n^2}{1/n^{1.5}} = \lim_{n \to \infty} \frac{\ln(n)}{n^{0.5}} \]

\[ H = \lim_{n \to \infty} \frac{1/n}{0.5n^{-0.5}} = \lim_{n \to \infty} \frac{2}{n^{1.5}} = 0 \]

Observe that \( \sum \frac{1}{n^{1.5}} \) converges by \( p \)-test \((p = 1.5 > 1)\). Since \( L = 0 \), \( \sum \frac{\ln(n)}{n^2} \) converges by LCT.

(d) We have a similar problem here.

\[ \frac{1}{n \ln(n)^2} \leq \frac{1}{n} \quad (\text{for } n \geq 3) \]

The inequality is in the wrong direction. Use Integral Test! Let \( f(x) = \frac{1}{x \ln(x)} \).

- \( f(x) \geq 0 \) (obvious)
- \( x \) is increasing, \( ln(x) \) is increasing
  - \( x \ln(x) \) is increasing
  - \( \frac{1}{x \ln(x)} \) is decreasing
\( f(x) \) is continuous (for \( n \geq 3 \))

Now we compute the integral.

\[
\int_2^\infty \frac{1}{x \ln(x)} \, dx = \int_{\ln(2)}^\infty \frac{1}{u} \, du = \ln(u) \bigg|_{\ln(2)}^\infty
\]

\( u = \ln(x) \), \( du = \frac{1}{x} \, dx \)

\[
= \lim_{R \to \infty} \ln(u) \bigg|_{\ln(2)}^R = \lim_{R \to \infty} \left( \ln(R) - \ln(\ln(2)) \right) = \infty
\]

Since the integral diverges, the series diverges by Integral Test.