Section 10.1: Sequences

Def: A sequence \( \{a_n\} \) is a function from some subset of \( \mathbb{N} \) to \( \mathbb{R} \). We call \( a_n \) the nth term and \( n \) the index.

We think of sequences as lists. Sequences can be described in several ways.

**Ex:**

<table>
<thead>
<tr>
<th>General Term</th>
<th>Domain</th>
<th>Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n = 1 - \frac{1}{n} )</td>
<td>( n \geq 1 )</td>
<td>( 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} ), ....</td>
</tr>
<tr>
<td>( a_n = 2^{-n} )</td>
<td>( n \geq 0 )</td>
<td>( 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} ), ....</td>
</tr>
<tr>
<td>( a_n = (-1)^n )</td>
<td>( n \geq 0 )</td>
<td>( 0, -1, 2, -3, 4, \ldots )</td>
</tr>
</tbody>
</table>

alternating factor

\( a_n \) is the nth digit in the decimal expansion of \( \pi \) (explicit formula not given)
Consider the sequence

\[ a_n = \frac{1}{2} \left( a_{n-1} + \frac{2}{a_{n-1}} \right) \]

with \( a_0 = 1 \). What is the limit of this sequence \( (n \to \infty) \)?

**Solution:**

Look at the first few terms:

\[ a_0 = 1 \]

\[ a_1 = \frac{1}{2} \left( a_0 + \frac{2}{a_0} \right) = 1.5 \]

\[ a_2 = \frac{1}{2} \left( a_1 + \frac{2}{a_1} \right) = \frac{17}{12} \approx 1.4167 \]
\[ a_3 = \frac{1}{2} \left( a_2 + \frac{2}{a_2} \right) = \frac{577}{408} \approx 1.414216 \]

It looks like \( a_n \) has limit \( \sqrt{2} \).

Let's suppose \( \lim_{n \to \infty} a_n = L \) (exists).

\[ a_n = \frac{1}{2} \left( a_{n-1} + \frac{2}{a_{n-1}} \right) \tag{*} \]

So take \( n \to \infty \) on both sides

\[ \lim_{n \to \infty} a_n = L , \quad \lim_{n \to \infty} a_{n-1} = L \]

\[ \{a_n\}_{n=1}^\infty = \frac{3}{2} a_1, a_2, \ldots \] same limit!
\[ \{a_{n-1}\}_{n=1}^\infty = \frac{3}{2} a_0, a_1, a_2, \ldots \]

So \( (*) \) becomes

\[ L = \frac{1}{2} \left( L + \frac{2}{L} \right) \implies L = \sqrt{2} \]

But how do we know the limit exists in the first place?

(Bonus: If you apply Newton's Method to the function \( f(x) = x^2 - 2 \) with initial guess 1, you recover \( \{a_n\} \).)
**Def:** We say \( \{a_n\} \) converges to a limit \( L \) if \( \lim_{n \to \infty} a_n = L \). We write this also as “\( a_n \to L \)” (“\( A\)-en goes to \( L \)”)

This means for all \( \varepsilon > 0 \), there exists \( N \) such that if \( n \geq N \), then \( |a_n - L| < \varepsilon \).

### Basic Terminology

**Boundedness**

- **bounded from below**

  \[
  m \leq a_n \\
  \text{for all } n
  \]

- **bounded from above**

  \[
  a_n \leq M \\
  \text{for all } n
  \]

- **bounded**

  \[
  m \leq a_n \leq M \\
  \text{for all } n
  \]

**Convergence**

\[
\begin{align*}
\text{converges} & \\
\lim_{n \to \infty} a_n & \exists \\
\text{diverges to } & \text{infinity} \\
\lim_{n \to \infty} a_n & = \infty \text{ OR } \\
\lim_{n \to \infty} a_n & = -\infty \\
\text{diverges} & \\
\lim_{n \to \infty} a_n & \text{ does not exist and is not } \pm \infty
\end{align*}
\]
• Monotonicity

<table>
<thead>
<tr>
<th></th>
<th>Monotonically Decreasing</th>
<th>Monotonically Increasing</th>
<th>Monotonic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a_{n+1} \leq a_n$</td>
<td>$a_{n+1} \geq a_n$</td>
<td>either mono. decreasing OR mono. increasing</td>
</tr>
</tbody>
</table>

Ex. 2

For each sequence, identify which properties above are satisfied.

**Sequence**

\[ a_n = 5 - \frac{1}{n} \]

**Solution**

\{ bounded from above, bounded \}

Converges

monotonically increasing

monotonic

\[ a_n = (-1)^n \]

\{ bounded from above, bounded \}

diverges
Some Basic Theorems

The following theorems establish some basic relationships among the various properties described above.

**Theorem 1:**
\[
\text{converges} \iff \text{bounded}
\]

**Theorem 2:**
\[
\text{monotonically increasing AND bounded from above} \implies \text{converges}
\]
Theorem 3:

monotonically decreasing AND bounded from below $\Rightarrow$ converges

--

Theorem 4:

Let $a_n = f(n)$. If $x \in \mathbb{R}$ and $\lim_{x \to \infty} f(x) = L$, then $\lim_{n \to \infty} a_n = L$ also.

Careful! (Theorem 4):

If $\lim_{x \to \infty} f(x)$ dne., theorem says nothing!

Ex: Let $f(x) = \sin(\pi x)$. Then $\lim_{x \to \infty} f(x)$ dne.

But $\{a_n\}_{n=0}^{\infty} = \{0, 0, \ldots\}$, so $a_n \to 0$.

Theorem 5: (Squeeze Theorem)

Suppose $a_n \to L$ and $b_n \to L$. If $a_n \leq c_n \leq b_n$ (for all $n$)

then $c_n \to L$ also.

Theorem 6:

If $|a_n| \to 0$, then $a_n \to 0$ also.
Proof:

Observe that $-|a_n| \leq a_n \leq |a_n|$. Now use Theorem 5 (Squeeze Theorem).

Careful! Theorem 6:

We must have $|a_n| \to 0$; any other limit does not allow the same conclusion. For instance, if $|a_n| \to 1$, then ${a_n}$ may or may not converge to 1.

Ex:

1. Let $a_n = 1$. Then $|a_n| \to 1$ and $a_n \to 1$
2. Let $a_n = (-1)^n$. Then $|a_n| = 1$ for all $n$.
   So $|a_n| \to 1$, but ${a_n}$ diverges.

Ex. 3

Calculate the limit of each sequence.

(a) $a_n = \left(1 + \frac{c}{n}\right)^n$  \quad (C = \text{const.})
(b) $a_n = C^{1/n}$  \quad (C = \text{const.} > 0)
(c) $a_n = n^{1/n}$

Solution:

By Theorem 4, we can find the limits using
Techniques of Calculus I (L'Hôpital's Rule)

(a) Let \( f(x) = \left(1 + \frac{c}{x}\right)^x \). If \( x \to \infty \), we have the expression "1^\infty". So put \( L = \lim_{x \to \infty} f(x) \).

\[
\ln(L) = \ln\left( \lim_{x \to \infty} \left(1 + \frac{c}{x}\right)^x \right)
= \lim_{x \to \infty} \left[ x \ln\left(1 + \frac{c}{x}\right) \right]
= \lim_{x \to \infty} \left[ \frac{\ln\left(1 + \frac{c}{x}\right)}{1/x} \right]
= \lim_{x \to \infty} \left[ \frac{1}{1 + \frac{c}{x}} \cdot \left(\frac{-c}{x^2}\right) \right]
= \lim_{x \to \infty} \left[ \frac{c}{1 + \frac{c}{x}} \right] = \frac{c}{1+0} = c
\]

So \( \ln(L) = c \), hence \( L = e^c \).

(For instance \( (1 - \frac{3}{n})^n \to e^{-3} \).)
(b) \[ \lim_{n \to \infty} C^{1/n} = C^0 = 1 \]

(c) Put \( L = \lim_{x \to \infty} x^{1/x} \). Now we have

\[
\ln(L) = \lim_{x \to \infty} \ln(x^{1/x}) = \lim_{x \to \infty} \frac{\ln(x)}{x} \]

\[
= \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0
\]

So \( \ln(L) = 0 \), hence \( L = 1 \).

(For instance, \( \lim_{n \to \infty} \sqrt[n]{n^3} = \lim_{n \to \infty} (n^{1/n})^3 = 1 \).)

\[\text{Ex. 4}\]

Calculate the limit of \( a_n = \frac{(-1)^n}{\sqrt{n}} \).

Solution:
We cannot go back to a real-variable function since \((-1)^x\) is ill-defined. Consider \( |a_n| \) instead.

\[|a_n| = \left| \frac{(-1)^n}{\sqrt{n}} \right| = \frac{|(-1)^n|}{\sqrt{n}} = \frac{1}{\sqrt{n}} \to 0\]
Since $|a_n| \to 0$, so does $a_n \to 0$.

**Geometric Sequence**

If $a_n = r^n$ (where $r$ is a constant), we call $\{a_n\}$ a geometric sequence with **common ratio** $r$.

The characterization of a geo. sequence:

\[
\frac{a_{n+1}}{a_n} = r = \text{const.} \quad (*)
\]

One more theorem...

**Theorem 7:**

If $a_n = r^n$ with $r = \text{const.}$, then

\[
\lim_{n \to \infty} a_n = \begin{cases} 
0 & \text{if } |r| < 1 \\
1 & \text{if } r = 1 \\
\infty & \text{if } r > 1 \\
\text{dne.} & \text{if } r \leq -1
\end{cases}
\]

**Proof:**

Let $f(x) = r^x$. Then from Calculus I,
\[ \lim_{{x \to \infty}} r^x = \begin{cases} 0 & \text{if } 0 \leq r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases} \]

If \( r < 0 \), this does not apply since \( f(x) \) is not well-defined, so we know nothing about \( a_n = r^n \). If \( -1 < r < 0 \), note that \( |r| < 1 \). So \( |r|^n \to 0 \). Hence by Theorem 6, \( r^n \to 0 \) also. If \( r < -1 \), the sequence \( r^n \) oscillates between larger and larger values of alternating sign. So \( \{r^n\} \) has no limit if \( r < -1 \). \( \blacksquare \)