Appendix: Complex Numbers

What should you know from Chapter 1?

- Suppose \( z = x + iy \) and \( w = u + iv \)
  - \( z + w \)
  - \( z - w \)
  - \( 2z, 3w \)
  - \( zw \)
  - \( \frac{z}{w} \)
  - \( \overline{z} = x - iy \) \hspace{1cm} \text{conjugate}
  - \( \text{Re}(z) = x \) \hspace{1cm} \text{real part}
  - \( \text{Im}(z) = y \) \hspace{1cm} \text{imaginary part}
  - \( |z| = \sqrt{x^2 + y^2} \) \hspace{1cm} \text{modulus}
  - \( |z| = \sqrt{z \overline{z}} \)

This is basic arithmetic we expect you already know. Now for Chapter 2...
There are three primary representations of complex numbers.

- **rectangular**: \(x + iy\)
- **polar**: \(r (\cos(\theta) + i \sin(\theta))\)
- **exponential**: \(re^{i\theta}\)

![Diagram of complex plane](image)

**complex plane**: \(x + iy \iff (x, y)\)

The polar form of a complex number is closely linked to polar coordinate:

\[ x + iy = r \cos(\theta) + i r \sin(\theta) \]

Note that \( r = \sqrt{x^2 + y^2} = |z| \)
Convention: We will usually allow \( \theta \)-values only in \([0, 2\pi]\).

The exponential form ties several topics in Calculus II together.

What does \( e^{3\theta} \) mean?

Let's look at the Taylor series for \( e^z \):

\[
e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}
\]

(valid for all real \( z \))

If \( e^{i\theta} \) is to make any sense at all, we should have

\[
e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}
\]

\[
i^0 = 1 \quad i^4 = i^3 \cdot i = (-i) \cdot i = 1
\]

\[
i^1 = i \quad i^5 = i
\]

\[
i^2 = -1 \quad i^6 = -1
\]

\[
i^3 = -i \quad i^7 = -i
\]
The powers of $i$ repeat in a cycle of $4$: $1, i, -1, -i$

Let's use this to rewrite the power series for $e^{i\theta}$!

$$e^{i\theta} = 1 + i\theta + \frac{i^2 \theta^2}{2!} + \frac{i^3 \theta^3}{3!} +$$

$$+ \frac{i^4 \theta^4}{4!} + \frac{i^5 \theta^5}{5!} + \frac{i^6 \theta^6}{6!} + \frac{i^7 \theta^7}{7!} + \ldots$$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i \theta^3}{3!} +$$

$$+ \frac{\theta^4}{4!} + i \frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i \theta^7}{7!} + \ldots$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \right) +$$

$$i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots \right)$$
Do you recognize these power series?

\[ e^{i\theta} = \cos(\theta) + i\sin(\theta) \]  

(Euler’s identity)

So the polar and exponential forms of a complex number are linked

\[ z = r\cos(\theta) + ir\sin(\theta) = re^{i\theta} \]

This makes multiplication/division easy

\[ e^{i\theta}e^{i\phi} = e^{i(\theta + \phi)} \]

\[ (e^{i\theta})^n = e^{in\theta} \]

(All usual rules of exponents still hold.)

Ex. 1

Find all forms of each complex number.

Solution:

(a) (Rect.) \[ z = -2\sqrt{2} + 2\sqrt{2}i \]
\[ r = \sqrt{(-2 \sqrt{2})^2 + (2 \sqrt{2})^2} = \sqrt{8 + 8} = \sqrt{16} = 4 \]

\[ \tan(\theta) = \frac{2 \sqrt{2}}{-2 \sqrt{2}} \quad \text{so} \quad \theta = \frac{3\pi}{4} \]

**Polar:** \[ z = 4 \cos\left(\frac{3\pi}{4}\right) + i \, 4 \sin\left(\frac{3\pi}{4}\right) \]

**Exponential:** \[ z = 4 \, e^{i \frac{3\pi}{4}} \]

\[ \text{(b) (Rect.)} \quad z = -1 \quad (-1 + 0i) \]

\[ r = 1 \quad \theta = \pi \]

**Polar:** \[ z = \cos(\pi) + i \, 1 \sin(\pi) \]

**Exponential:** \[ z = 1 \, e^{i \pi} \]

\[ \text{(c) (Rect.)} \quad z = i \quad (0 + 1i) \]

\[ r = 1 \quad \theta = \frac{\pi}{2} \]

**Polar:** \[ z = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \]

**Exponential:** \[ z = e^{i \frac{\pi}{2}} \]
DeMoivre's Formula

\[(e^{i\theta})^n = e^{in\theta}\]

\[(\cos \theta + i\sin \theta)^n = \cos(n\theta) + i\sin(n\theta)\]

This helps with trigonometric identities.

**Ex. 2**

Use DeMoivre's Formula to derive a triple angle formula for \(\sin(3\theta)\) and \(\cos(3\theta)\).

**Solution**: We have that

\[(\cos \theta + i\sin \theta)^3 = \cos(3\theta) + i\sin(3\theta) \quad (1)\]

Expand the left side using algebra.

\[(\cos \theta + i\sin \theta)^3 = \cos^3 \theta + 3\cos^2 \theta \cdot i\sin \theta +
\quad + 3\cos \theta \cdot i^2 \sin^2 \theta + i^3 \sin^3 \theta\]

\[= [\cos^3 \theta - 3\cos \theta \sin^2 \theta] + i [3\cos^2 \theta \sin \theta - \sin^3 \theta] \]
Now back to Equation (1):

\[
\left[ \cos^3 \theta - 3 \cos \theta \sin^2 \theta \right] + i \left[ 3 \cos^2 \theta \sin \theta - \sin^3 \theta \right]
\]

\[= \cos(3\theta) \quad = \sin(3\theta)\]

So we obtain the formulas:

\[
\cos(3\theta) = \cos(\theta)^3 - 3\cos(\theta)\sin(\theta)^2
\]
\[
\sin(3\theta) = 3\cos(\theta)^2 \sin(\theta) - \sin(\theta)^3
\]

Using \( \cos(\theta)^2 + \sin(\theta)^2 = 1 \), these formulas are more commonly written as:

\[
\cos(3\theta) = 4\cos(\theta)^3 - 3\cos(\theta)
\]
\[
\sin(3\theta) = 3\sin(\theta) - 4\sin(\theta)^3
\]

**Finding Complex Roots**

Our primary goal now is to solve equations of the form

\[ z^n = w \]

where \( n > 0 \) is a given integer and \( w \) is a given complex number. In general,
there are \( n \) distinct solutions for \( z \). We say \( z \) is a \( n \)th root of \( w \). We will see that one complication is that the function \( f(z) = e^z \) is not one-to-one for complex numbers.

There is no such thing as “\( \ln \)” for complex numbers!

**Theorem #1:**

Suppose \( e^z = 1 \). Then there is an integer \( N \) such that \( z = 2\pi i N \).

**Proof:**

Let \( z = x + i y \) with \( x \) and \( y \) real.

\[
1 = e^z = e^{x+iy} = e^x e^{iy}
\]

Now take the modulus of both sides

\[
1 = |1| = |e^x e^{iy}| = |e^x| |e^{iy}| = e^x |e^{iy}|
\]

Note that \( e^{iy} = \cos(y) + i \sin(y) \). So…
\[ |e^{iy}| = \sqrt{\cos(y)^2 + \sin(y)^2} = 1 \]

So now we have:

\[ 1 = e^x \]

Since \( x \) is real, we have \( x = 0 \), and we are left with

\[ 1 = e^{iy} = \cos(y) + i\sin(y) \]

This is equivalent to the system:

\[ \cos(y) = 1 \]
\[ \sin(y) = 0 \]

The solutions to this system are \( y = 2\pi N \) where \( N \) is any integer. Hence we conclude \( z = iy = 2\pi i N \).

This theorem explicitly shows that \( e^z \) is not one-to-one. How do we solve \( e^z = e^w \)?

**Theorem #2**

Suppose \( e^z = e^w \). Then there is an integer \( N \) such that \( z = w + 2\pi i N \).
If \( e^z = e^w \), then \( e^{z-w} = 1 \) by usual rules of exponents. From Theorem \#1, we have
\[
z - w = 2\pi i N
\]
for some integer \( N \). So \( z = w + 2\pi i N \).

Theorem \#2 is crucial in finding roots of complex numbers.

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**Ex. 2**

Find all solutions to \( z^3 = 8i \).

**Solution:**

First write both sides of the equation in exponential form. Suppose
\[
z = re^{i\theta}
\]
where \( r \) and \( \theta \) are unknown. Now write \( 8i \) in exponential form:
\[
8i = Re^{i\Phi}
\]
The equation \( z^3 = 8i \) is now written as:
\[
(z e^{i\theta})^3 = 8 e^{i\frac{\pi}{2}}
\]

This splits into two equations:
\[
\begin{align*}
\Gamma^3 &= 8 \\
\Gamma &= 2 \\
i \cdot 3\theta &= i \frac{\pi}{2} + 2\pi i N
\end{align*}
\]

(\text{Theorem \#2})

Note the use of Theorem \#2! So now
\[
\theta = \frac{\pi}{6} + \frac{2\pi}{3} \cdot N
\]

So all solutions have the form:
\[ z = r e^{i \theta} = 2 e^{i \left( \frac{\pi}{6} + \frac{2\pi}{3} N \right)} \]

\[ = 2 \cos \left( \frac{\pi}{6} + \frac{2\pi}{3} N \right) + i \cdot 2 \sin \left( \frac{\pi}{6} + \frac{2\pi}{3} N \right) \]

where \( N \) is an integer. This looks like there are infinitely many solutions. But the periodicity of cosine and sine implies there are only 3 distinct solutions.

\( N = 0: \quad z_1 = 2 e^{i \frac{\pi}{6}} = 2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \]

\[ = \sqrt{3} + i \]

\( N = 1: \quad z_2 = 2 e^{i \frac{5\pi}{6}} = 2 \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) \]

\[ = \sqrt{3} - i \]

\( N = 2: \quad z_3 = 2 e^{i \frac{9\pi}{6}} = 2 \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) \]

\[ = 2 (0 - i) = -2i \]

\[ \text{Ex. 3} \]

Find all complex solutions to

\[ z^3 = -2 - 2\sqrt{3} i \]
Solution:

Let \( z = re^{i\theta} \) (\( r, \theta \) unknown). Convert \( w = -2 - 2\sqrt{3}i \) to exponential form.

Now substitute into our equation.

\[
\begin{align*}
|w| &= \sqrt{(-2)^2 + (-2\sqrt{3})^2} = 4 \\
\tan(\varphi) &= \frac{-2\sqrt{3}}{-2} = \sqrt{3} \\
\varphi &= \frac{4\pi}{3}
\end{align*}
\]

\[
\begin{align*}
z^3 &= -2 - 2\sqrt{3}i \\
(r e^{i\theta})^3 &= 4 e^{i\frac{4\pi}{3}} \\
r^3 e^{i3\theta} &= 4 e^{i\frac{4\pi}{3}} \\
r^3 &= 4 \\
\text{i}3\theta &= i\frac{4\pi}{3} + 2\pi i N \\
\theta &= \frac{4\pi}{3} + \frac{2\pi}{3} N
\end{align*}
\]
So our three unique solutions are

\[ N = 0 : \quad z_1 = 4^{1/3} e^{i \frac{4\pi}{9}} = 4^{1/3} \left( \cos \frac{4\pi}{9} + i \sin \frac{4\pi}{9} \right) \]

\[ N = 1 : \quad z_2 = 4^{1/3} e^{i \frac{10\pi}{9}} = 4^{1/3} \left( \cos \frac{10\pi}{9} + i \sin \frac{10\pi}{9} \right) \]

\[ N = 2 : \quad z_3 = 4^{1/3} e^{i \frac{16\pi}{9}} = 4^{1/3} \left( \cos \frac{16\pi}{9} + i \sin \frac{16\pi}{9} \right) \]

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**Geometric Interpretation of Multiplication**

Consider the following:

\[ z_1 = r_1 e^{i \theta_1}, \quad z_2 = r_2 e^{i \theta_2} \]

\[ z_1 z_2 = r_1 r_2 e^{i (\theta_1 + \theta_2)} \]

So lengths multiply and angles add.
If $\omega^n = 1$, we say $\omega$ is an \textit{nth root of unity}. What do these solutions look like? (Let $\omega = r e^{i\theta}$.)

$$\omega^n = 1$$
$$\gamma^n e^{i\theta} = 1 \cdot e^{i0}$$

$$\gamma^n = 1$$
$$e^{i\theta} = e^{i0}$$

$$\gamma = 1$$
$$i\theta = 0 + 2\pi i N$$

where $N$ is an integer $\rightarrow \theta = 2\pi \frac{N}{n}$

Hence the $n$ \textit{nth roots of unity} are:

$N=0$: $\omega_1 = 1$

$N=1$: $\omega_2 = \cos (2\pi \cdot \frac{1}{n}) + i \sin (2\pi \cdot \frac{1}{n})$

$N=2$: $\omega_3 = \cos (2\pi \cdot \frac{2}{n}) + i \sin (2\pi \cdot \frac{2}{n})$

$\vdots$

$N=n-1$: $\omega_n = \cos (2\pi \cdot \frac{n-1}{n}) + i \sin (2\pi \cdot \frac{n-1}{n})$
Note that $|\omega_k| = 1$ for all $k$ and there are $n$ solutions, each separated by the angle $2\pi/n$. So what if we plot these points in the complex plane?

In general, the $n$th roots of unity are the vertices of a regular $n$-gon inscribed in the unit circle, with one vertex at $w = 1$.

Finally, let $\omega_1 = e^{i \frac{2\pi}{N}}$. Then by rules
of exponents:
\[ w_k = e^{i \frac{2\pi k}{n}} = (e^{i \frac{2\pi}{n}})^k = \omega_1^k \]

If you know one root of unity (not 1), you get the others by higher powers:

\[ 1, \omega_1, \omega_1^2, \omega_1^3, \ldots, \omega_1^{n-1} \]

The \( n \) \( n \)th roots of unity

This offers an alternative way of finding complex roots. Suppose we want to solve:

\[ z^n = w \]

Let \( z_0 \) be any solution to this equation. That is, \( z_0^n = w \). Now let \( \omega \) be an \( n \)th root of unity.

\[ \omega^n = 1 \]

Then \( z = \omega z_0 \) solves our equation!

\[ (\omega z_0)^n = \omega^n z_0^n = 1 \cdot w = w \]
So if you can find any solution to \( z^n = w \) and any \( n \)th root of unity \( \omega \), the \( n \) solutions to \( z^n = w \) are:

- **Solution 1:** \( z_1 = z_0 \)
- **Solution 2:** \( z_2 = \omega z_0 \)
- **Solution 3:** \( z_3 = \omega^2 z_0 \)
- **Solution 4:** \( z_4 = \omega^3 z_0 \)
- \( \vdots \)
- **Solution \#n:** \( z_n = \omega^{n-1} z_0 \)

**Ex:**

Given that \( (1+i)^7 = 8-8i \), find all solutions to \( z^7 = 8-8i \).

**Solution:**

Let \( \omega \) be any 7th root of unity except 1. One choice is

\[ \omega = e^{i \frac{2\pi}{7}} \]

(Check: \( \omega^7 = (e^{i \frac{2\pi}{7}})^7 = e^{i2\pi} = \cos(2\pi) + i\sin(2\pi) = 1 \))
One solution to \( z^7 = 8 - 8i \) is

\[ z_0 = 1 + i \]

Thus the 7 solutions to \( z^7 = 8 - 8i \) are:

\[ z_1 = z_0 = 1 + i \]

\[ z_2 = \omega z_0 = \left( \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7} \right)(1 + i) \]

\[ z_3 = \omega^2 z_0 = \left( \cos \frac{4\pi}{7} + i \sin \frac{4\pi}{7} \right)(1 + i) \]

\[ z_4 = \omega^3 z_0 = \ldots \]

\[ z_5 = \omega^4 z_0 = \ldots \]

\[ z_6 = \omega^5 z_0 = \ldots \]

\[ z_7 = \omega^6 z_0 = \ldots \]