Math 152:

Calculus II for Engineering

Lecture Notes by Dr. G

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Section 5.7: Substitution Method

* Basic substitution example:

\[
\int x \cos(x^2) \, dx
\]

**constant multiple of derivative of inside function is sitting outside.**

<table>
<thead>
<tr>
<th>( u = x^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( du = 2x , dx )</td>
</tr>
<tr>
<td>( dx = \frac{du}{2x} )</td>
</tr>
</tbody>
</table>

Translation table from \( x \) to \( u \).

\[
\int x \cos(x^2) \, dx = \int x \cos(u) \cdot \frac{du}{2x} = \int \frac{1}{2} \cos(u) \, du = \frac{1}{2} \sin(u) + C = \frac{1}{2} \sin(x^2) + C
\]

* Substitution can also be used to get around difficult algebra or transform the integrand into something for which the method of
Integration is more obvious.

**Ex. 1**

Calculate \( \int x \sqrt{1+x} \, dx \)

**Solution:**

Note: \( \sqrt{1+x} \neq \sqrt{1} + \sqrt{x} \)

So use substitution to get around this.

\[
\begin{align*}
  u &= 1 + x \\
  du &= dx
\end{align*}
\]

\[
\int x \sqrt{1+x} \, dx = \int x \sqrt{u} \, du
\]

What do we do with this?

Use the substitution to solve for \( x \).

\[
\begin{align*}
  u &= 1 + x \\
  x &= u - 1
\end{align*}
\]

\[
\int x \sqrt{1+x} \, dx = \int (u-1) u^{1/2} \, du
\]

\[
= \int \left( u^{3/2} - u^{1/2} \right) \, du = \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} + C
\]

\[
= \frac{2}{5} (x+1)^{5/2} - \frac{2}{3} (x+1)^{3/2} + C
\]
Ex. 2  

Calculate \( \int_0^{\sqrt{e}-1} \frac{x^3}{x^2+1} \, dx \)

**Solution:**

Similar to last example.

\[
\begin{align*}
\text{Let } u &= x^2 + 1 \\
du &= 2x \, dx \\
dx &= \frac{du}{2x}
\end{align*}
\]

\[
\int_0^{\sqrt{e}-1} \frac{x^3}{x^2+1} \, dx = \int_1^e \frac{x^3}{u} \cdot \frac{du}{2x}
\]

\[
= \int_1^e \frac{1}{2} \cdot \frac{x^2}{u} \, du = \frac{1}{2} \int_1^e \frac{u-1}{u} \, du
\]

\[
= \frac{1}{2} \int_1^e (1 - \frac{1}{u}) \, du = \frac{1}{2} \left( u - \ln(u) \right) \bigg|_1^e
\]

\[
= \frac{1}{2} (e - 1) - \frac{1}{2} (1 - 0) = \frac{1}{2} e - 1
\]
Calculate \( \int \frac{1}{(1+\sqrt{x})^3} \, dx \).

**Solution:**

Only obvious substitution is

\[
u = 1 + \sqrt{x}
\]

\[
du = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} \, dx
\]

Better idea: solve for \( x \) (if you can), then differentiate. This gives \( dx \) directly in terms of \( x \).

\[
u = 1 + \sqrt{x}
\]

\[
(u-1)^2 = x
\]

\[
2(u-1) \, du = dx
\]

\[
\int \frac{1}{(1+\sqrt{x})^3} \, dx = \int \frac{1}{u^3} \cdot 2(u-1) \, du
\]

\[
= \int \left( \frac{2}{u^2} - \frac{2}{u^3} \right) \, du = -\frac{2}{u} + \frac{1}{u^2} + C
\]

\[
= \frac{-2}{1+\sqrt{x}} + \frac{1}{(1+\sqrt{x})^2} + C
\]
Alternatively, we can use partial fractions (once we learn that method).

\[ u = \sqrt{x} \]
\[ u^2 = x \]
\[ 2udu = dx \]

\[ \int \frac{1}{(1+\sqrt{x})^3} \, dx = \int \frac{2u}{(1+u)^3} \, du \]

The substitution has transformed the integrand into a rational function, which can be integrated using the method of partial fractions.

**Ex. 4**

Calculate each of the following:

(a) \[ \int \frac{2x + 1}{x^2 + x} \, dx \]

(b) \[ \int \frac{2x + 2}{x^2 + x} \, dx \]

(c) \[ \int \frac{2x + 3}{x^2 + x} \, dx \]
Solution:

Note that all integrands are similar, up to the constant term in the numerator.

(a) Basic substitution:

\[
\begin{align*}
    u &= x^2 + x \\
    du &= (2x+1)\,dx
\end{align*}
\]

\[
\int \frac{2x+1}{x^2+x}\,dx = \int \frac{1}{u}\,du = \ln |x^2+x| + C
\]

(b) Basic algebra:

\[
\int \frac{2x+2}{x^2+x}\,dx = \int \frac{2(x+1)}{x(x+1)}\,dx = \int \frac{2}{x}\,dx
\]

\[= 2 \ln |x| + C\]

(c) Algebra? Substitution?

\[
\int \frac{2x+3}{x^2+x}\,dx = \int \frac{2x+2+1}{x^2+x}\,dx
\]

\[= \int \frac{2x+2}{x^2+x}\,dx + \int \frac{1}{x^2+x}\,dx\]
= 2 \ln |x| + \int \frac{1}{x^2 + x} \, dx

How can we handle this? Our methods so far are not sufficient. We need partial fractions.

The method of partial fractions lets us integrate rational functions by decomposing them into sums that are easily integrated:

\[
\frac{1}{x^2 + x} = \frac{1}{x} - \frac{1}{x+1}
\]

We will learn the algebra to “un-simplify” rational functions.

Ex. 5

Calculate \( \int \tan(x) \sec(x)^2 \, dx \).

Solution:
- $u = \tan(x)$
  $\,du = \sec(x)^2 \,dx$

\[
\int \tan(x)\sec(x)^2 \,dx = \int u \,du \\
= \frac{1}{2} u^2 + C = \frac{1}{2} \tan(x)^2 + C
\]

- $u = \sec(x)$
  $\,du = \sec(x)\tan(x) \,dx$

\[
\int \tan(x)\sec(x)^2 \,dx = \int u \,du \\
= \frac{1}{2} u^2 + C = \frac{1}{2} \sec(x)^2 + C
\]

Are both really correct? What's going on?

**Note:** \(\sec(x)^2 = \tan(x)^2 + 1\)

So the issue is that the "+ C" means a different constant for the two solutions!

* Getting multiple correct forms of an antiderivative is very common!
Section 7.1: Integration-by-parts

* Substitution rule = backwards chain rule
* Integration-by-parts = backwards product rule

\[
\frac{d}{dx} (uv) = u \frac{dv}{dx} + \frac{du}{dx} v
\]

\[
u \frac{dv}{dx} = \frac{d}{dx} (uv) - \frac{du}{dx} v
\]

\[
\int u \, dv = uv - \int v \, du
\]

\[
\int_a^b u \, dv = uv \bigg|_a^b - \int_a^b v \, du
\]

**Ex. 1**

Calculate \( \int x \cos(x) \, dx \)

**Solution:**
\[ \int x \cos(x) \, dx \]

1. Choose \( dv \) so that \( v \) is easy to calculate.
2. Choose \( u \) so that \( du \) is algebraically simpler.

So we choose:

\[
\begin{align*}
    u &= x \\
    dv &= \cos(x) \, dx \\
    du &= dx \\
    v &= \sin(x)
\end{align*}
\]

IBP table

\[
\int u \, dv = uv - \int v \, du
\]

\[
\int x \cos(x) \, dx = x \sin(x) - \int \sin(x) \, dx
\]

\[
\begin{align*}
    &= x \sin(x) + \cos(x) + C
\end{align*}
\]

Ex. 2
Calculate \( \int xe^x \, dx \).

**Solution:**

\[
\begin{align*}
\begin{array}{c}
u = x \\
\frac{du}{dx} = dx
\end{array}
\quad \begin{array}{c}
dv = e^x \, dx \\
v = e^x
\end{array}
\end{align*}
\]

\[
\int u \, dv = uv - \int v \, du
\]

\[
\int xe^x \, dx = xe^x - \int e^x \, dx
\]

\[
= xe^x - e^x + C
\]

What if we made the opposite choice for \( u \) and \( du \)?

\[
\begin{align*}
\begin{array}{c}
u = e^x \\
\frac{du}{dx} = e^x \, dx
\end{array}
\quad \begin{array}{c}
dv = x \, dx \\
v = \frac{1}{2} x^2
\end{array}
\end{align*}
\]

\[
\int xe^x \, dx = \frac{1}{2} x^2 e^x - \int \frac{1}{2} x^2 e^x \, dx
\]
Ex. 3

Calculate \( \int_0^\pi x^3 \cos(x) \, dx \).

Solution:

\[
\begin{align*}
  u &= x^3 & dv &= \cos(x) \, dx \\
  du &= 3x^2 \, dx & v &= \sin(x)
\end{align*}
\]

\[
\int_0^\pi x^3 \cos(x) \, dx =
\]

\[
= x^3 \sin(x) \bigg|_0^\pi - \int_0^\pi 3x^2 \sin(x) \, dx
\]

\[
= 0
\]

\[
= - \int_0^\pi 3x^2 \sin(x) \, dx
\]

\[
\begin{align*}
  u &= 3x^2 & dv &= -\sin(x) \, dx \\
  du &= 6x \, dx & v &= \cos(x)
\end{align*}
\]
\[- \int_0^\pi 3x^2 \sin(x) \, dx = \]
\[= 3x^2 \cos(x) \bigg|_0^\pi - \int_0^\pi 6x \cos(x) \, dx \]
\[= (3\pi^2 \cdot (-1) - 0) \]
\[= -3\pi^2 \]
\[= -3\pi^2 - \int_0^\pi 6x \cos(x) \, dx \]  
Need IBP one more time!

Is there an easier way?

* Tabular Integration
  1. one factor must be a polynomial
  2. other factor must be easily antidifferentiated multiple times
\[ \int x^3 \cos(x) \, dx = \]

\[ = x^3 \sin(x) + 3x^2 \cos(x) - 6x \sin(x) - 6 \cos(x) + C \]

\[ \int_0^\pi x^3 \cos(x) \, dx = \left( \ldots \right) \big|_0^\pi = \ldots \]
Calculate \( \int \tan^{-1}(x) \, dx \)

Solution:

\[
\begin{align*}
\text{Let } u &= \tan^{-1}(x) & du &= \frac{1}{1+x^2} \, dx \\
\text{Then } v &= x
\end{align*}
\]

\[
\int \tan^{-1}(x) \, dx = x \tan^{-1}(x) - \int \frac{x}{1+x^2} \, dx
\]

Now use substitution:

\[
\begin{align*}
\text{Let } u &= 1 + x^2 & \frac{1}{2} \, du &= x \, dx
\end{align*}
\]

\[
\int \tan^{-1}(x) \, dx = x \tan^{-1}(x) - \frac{1}{2} \int \frac{1}{u} \, du
\]

\[
= x \tan^{-1}(x) - \frac{1}{2} \ln (1+x^2) + C
\]
Ex. 5

Calculate \[ \int \frac{x^3}{\sqrt{x^2+1}} \, dx \] .

Solution:

(a) Substitution:
\[
\begin{align*}
  u &= x^2 + 1 \\
  \frac{1}{2} \, du &= x \, dx
\end{align*}
\]

\[
\int \frac{x^2 \cdot x}{\sqrt{x^2+1}} \, dx = \int \frac{(u-1)}{\sqrt{u}} \cdot \frac{1}{2} \, du
\]

\[
= \frac{1}{2} \int (u^{1/2} - u^{-1/2}) \, du
\]

\[
= \frac{1}{2} \left( \frac{2}{3} u^{3/2} - 2 u^{1/2} \right) + C
\]

\[
= \frac{1}{3} (1+x^2)^{3/2} - (1+x^2)^{1/2} + C
\]

(b) Integration by parts:
\[
\int \frac{x^3}{\sqrt{x^2+1}} \, dx
\]

\[u = x^2 \quad \text{substitution rule}\]
\[du = 2x \, dx \quad v = \sqrt{x^2+1}\]
\[
\frac{dv}{dx} = \frac{1}{2} \cdot (x^2+1)^{-\frac{1}{2}} \cdot 2x
\]

\[
\int \frac{x^3}{\sqrt{x^2+1}} \, dx = x^2 \sqrt{x^2+1} - \int 2x \sqrt{x^2+1} \, dx
\]

\[= x^2 (x^2+1)^{\frac{1}{2}} - \frac{2}{3} (x^2+1)^{\frac{3}{2}} + C
\]

Ex. 6

Calculate \( \int 3x^2 \tan^{-1}(x) \, dx \).

Solution:
\[u = \tan^{-1}(x) \quad dv = 3x^2 \, dx\]
\[ du = \frac{1}{1 + x^2} \, dx \quad v = x^3 \]

\[
\int 3x^2 \tan^{-1}(x) \, dx =
\]

\[
= x^3 \tan^{-1}(x) - \int \frac{x^3}{1 + x^2} \, dx
\]

(a) Substitution:

\[
\begin{aligned}
& u = x^2 + 1 \\
& \frac{1}{2} \, du = x \, dx
\end{aligned}
\]

\[
\int \frac{x^3}{1 + x^2} \, dx = \frac{1}{2} \int \frac{(u-1)}{u} \, du
\]

\[
= \frac{1}{2} \int \left(1 - \frac{1}{u}\right) \, du = \frac{1}{2} u - \frac{1}{2} \ln(u) + C
\]

\[
= \frac{1}{2} \left(1 + x^2\right) - \frac{1}{2} \ln \left(1 + x^2\right) + C
\]

(b) Algebra, then substitution

\[
\frac{x^3}{1 + x^2} = x - \frac{x}{1 + x^2} \quad \text{**polynomial long division**}
\]
\[
\int \frac{x^3}{1+x^2} \, dx = \int \left( x - \frac{x}{1+x^2} \right) \, dx \\
= \frac{1}{2} x^2 - \frac{1}{2} \ln(1+x^2) + C
\]

**Ex. 7**

Calculate \( \int e^{2x} \cos(x) \, dx \).

**Solution**:

\[
\begin{align*}
\quad u &= \cos(x) & du &= e^{2x} \, dx \\
\quad dv &= e^{2x} \, dx & v &= \frac{1}{2} e^{2x} \\
\quad du &= -\sin(x) \, dx & v &= \frac{1}{2} e^{2x}
\end{align*}
\]

\[
\int e^{2x} \cos(x) \, dx = \frac{1}{2} e^{2x} \cos(x) + \int \frac{1}{2} e^{2x} \sin(x) \, dx
\]

\[
\begin{align*}
\quad u &= \sin(x) & dv &= \frac{1}{2} e^{2x} \, dx \\
\quad du &= \cos(x) \, dx & v &= \frac{1}{4} e^{2x}
\end{align*}
\]
\[ \int e^{2x} \cos(x) \, dx = \frac{1}{2} e^{2x} \cos(x) + \frac{1}{4} e^{2x} \sin(x) - \frac{1}{4} \int e^{2x} \cos(x) \, dx \]

Let \( I = \int e^{2x} \cos(x) \, dx \). Then:

\[ I = \frac{1}{2} e^{2x} \cos(x) + \frac{1}{4} e^{2x} \sin(x) - \frac{1}{4} I \]

Algebraically solve for \( I \):

\[ \Rightarrow I = \frac{4}{5} \left( \frac{1}{2} e^{2x} \cos(x) + \frac{1}{4} e^{2x} \sin(x) \right) + C \]

Ex. 8

(Reduction Formula)

Calculate \( \int_1^e (\ln x)^5 \, dx \) by first finding a reduction formula for \( \int_1^e (\ln x)^n \, dx \).

Solution:
Let \( I_n = \int_1^e (\ln x)^n \, dx \). Use integration by parts.

\[
\begin{align*}
u &= (\ln x)^n & dv &= dx \\
du &= n (\ln x)^{n-1} \, dx & v &= x
\end{align*}
\]

\[
I_n = x (\ln x)^n \bigg|_1^e - \int_1^e n (\ln x)^{n-1} \, dx
\]

\[
= n \int_1^e (\ln x)^{n-1} \, dx = n \cdot I_{n-1}
\]

\[
I_n = (e \cdot 1^n - 1 \cdot 0) - n I_{n-1}
\]

So we have...

\[
I_n = e - n I_{n-1}
\]

\( \rightarrow \) reduction formula

To calculate \( \int_1^e (\ln x)^5 \, dx \), we use
Our reduction formula:

\[ I_5 = e - 5I_4 \]
\[ I_4 = e - 4I_3 \]
\[ I_3 = e - 3I_2 \]
\[ I_2 = e - 2I_1 \]
\[ I_1 = e - 1 \cdot I_0 \quad \text{stop!} \]

So what is \( I_0 \)?

\[ I_0 = \int_1^e (\ln x)^0 \, dx = \int_1^e 1 \, dx = e - 1 \]

Now back substitute:

\[ I_0 = e - 1 \]
\[ I_1 = e - (e - 1) = 1 \]
\[ I_2 = e - 2 \]
\[ I_3 = e - 3(e - 2) = 6 - 2e \]
\[ I_4 = e - 4(6 - 2e) = 9e - 24 \]
\[ I_5 = e - 5(9e - 24) = 120 - 44e \]
Ex. 9

Calculate \( \int_{1}^{4} e^{\sqrt{x}} \, dx \).

Solution:
First use substitution.

\[
\begin{align*}
    u &= \sqrt{x} \\
    u^2 &= x \\
    2u \, du &= dx
\end{align*}
\]

\[
\int_{1}^{4} e^{\sqrt{x}} \, dx = \int_{1}^{2} 2ue^{u} \, du
\]

use IBP! (tabular)

\[
\begin{array}{c}
    2u \\
    2 \\
    0
\end{array}
\]

\[
\begin{array}{c}
    e^{u} \\
    e^{u} \\
    e^{u}
\end{array}
\]

\[
\int_{1}^{2} 2ue^{u} \, du = (2ue^{u} - 2e^{u}) \bigg|_{1}^{2} = \ldots
\]
Priority for choosing $u$ in IBP:

$L$: logarithm
$I$: inverse trigonometric
$A$: algebraic
$T$: trigonometric
$E$: exponential
Section 7.2: Trigonometric Integrals

- ∫ sin(x)^m cos(x)^n dx
- ∫ tan(x)^m sec(x)^n dx
- ∫ cot(x)^m csc(x)^n dx

Useful identities:
- \( \sin(x)^2 = \frac{1}{2} - \frac{1}{2} \cos(2x) \)
- \( \cos(x)^2 = \frac{1}{2} + \frac{1}{2} \cos(2x) \)
- \( \cos(x)^2 + \sin(x)^2 = 1 \)
- \( 1 + \tan(x)^2 = \sec(x)^2 \)
- \( \cot(x)^2 + 1 = \csc(x)^2 \)

* We will look at a few types of each integral and discuss the various strategies.

\(\rightarrow\) Note: See end of notes for summary!
Ex. 1

Calculate \( \int \sin(x)^3 \, dx \)

Solution:

\[
\int \sin(x)^3 \, dx = \int \sin(x)^2 \cdot \sin(x) \, dx
\]

\[
= \int \left(1 - \cos(x)^2\right) \cdot \sin(x) \, dx
\]

\[
u = \cos(x), \quad -du = \sin(x) \, dx
\]

\[
= \int (1 - u^2) \cdot (-du) = -u + \frac{u^3}{3} + C
\]

\[
= -\cos(x) + \frac{\cos(x)^3}{3} + C
\]

Ex. 2

Calculate \( \int \frac{\sin(x)^5}{\cos(x)^3} \, dx \)

Solution:

\[
\int \frac{\sin(x)^5}{\cos(x)^3} \, dx = \int \frac{\sin(x)^4}{\cos(x)^3} \cdot \sin(x) \, dx
\]
\[= \int \frac{(\sin(x)^2)^2}{\cos(x)^3} \sin(x) \, dx\]

\[= \int \frac{(1-\cos(x)^2)^2}{\cos(x)^3} \sin(x) \, dx\]

\[u = \cos(x), \quad -du = \sin(x) \, dx\]

\[-\int \frac{(1-u^2)^2}{u^3} \, du = \int \frac{u^4 - 2u^2 + 1}{u^3} \, du\]

\[-\int (u - \frac{2}{u} + \frac{1}{u^3}) \, du\]

\[-\left(\frac{u^2}{2} - 2 \ln|u| - \frac{1}{2u^2}\right) + C\]

\[-\left(\frac{\cos(x)^2}{2} - 2 \ln|\cos(x)| - \frac{1}{2\cos(x)^2}\right) + C\]

---

**Ex. 3**

Calculate \(\int \sin(x)^4 \cos(x)^5 \, dx\).

**Solution:**
\[
\int \sin(x)^4 \cos(x)^5 \, dx = \int \sin(x)^4 \cos(x)^4 \cos(x) \, dx \\
= d\left(\sin(x)\right) \\
= \int \sin(x)^4 \left(1-\sin(x)^2\right)^2 \cos(x) \, dx \\
= \cos(x)^2 \\
\]

\[u = \sin(x) \Rightarrow du = \cos(x) \, dx\]

\[= \int u^4 \left(1-u^2\right)^2 \, du \]

\[= \int u^4 (u^4 - 2u^2 + 1) \, du = \int (u^8 - 2u^6 + u^4) \, du \]

\[= \frac{u^9}{9} - \frac{2u^7}{7} + \frac{u^5}{5} + C \]

\[= \frac{\sin(x)^9}{9} - \frac{2\sin(x)^7}{7} + \frac{\sin(x)^5}{5} + C \]

---

**Ex. 4**

Calculate \[\int \sin(x)^2 \, dx\].

**Solution:**
(a) Double-angle formula.
\[
\int \sin^2(x) \, dx = \int \left( \frac{1}{2} - \frac{1}{2} \cos(2x) \right) \, dx
\]
\[
= \frac{1}{2} x - \frac{1}{4} \sin(2x) + C
\]

(b) Integration by parts:
\[
u = \sin(x)^2 - 1 \quad dv = \sin(x) \, dx
\quad du = \cos(x) \, dx \quad v = -\cos(x)
\]
\[
\int u \, dv = uv - \int v \, du
\]
\[
\int \sin^2(x) \, dx = -\sin(x) \cos(x) + \int \cos^2(x) \, dx
\]

This integral always has a factor of \(\cos(x)^2\), which we rewrite as \(1 - \sin(x)^2\).
\[
\int \sin^2(x) \, dx = -\sin(x) \cos(x) + \int (1 - \sin^2(x)) \, dx
\]

Let \(I = \int \sin^2(x) \, dx\).
\[ I = -\sin(x) \cos(x) + \int 1 \, dx - I \]
\[ 2I = -\sin(x) \cos(x) + x + C \]

\[ I = -\frac{1}{2} \sin(x) \cos(x) + \frac{1}{2} x + C \]

**Ex. 5**

Calculate \( \int \sin(x)^2 \cos(x)^2 \, dx \).

**Solution**:

First rewrite the integrand in terms of \( \sin(x) \).

\[
\int \sin(x)^2 \cos(x)^2 \, dx = \int \sin(x)^2 (1-\sin(x)^2) \, dx
\]

\[
= \int \sin(x)^2 \, dx - \int \sin(x)^4 \, dx
\]

\[ \text{deal with each integral, which should be just a pure even power of } \sin(x). \]

We will focus on \( \int \sin(x)^4 \, dx \).
Use integration by parts:

\[ u = \sin(x)^3 \quad dv = \sin(x) \, dx \]
\[ du = 3 \sin(x)^2 \cos(x) \, dx \quad v = -\cos(x) \]

\[
\int \sin(x)^4 \, dx = -\sin(x)^3 \cos(x) + \int 3\sin(x)^2 \cos(x)^2 \, dx
\]

This integral must have a factor of \( \cos(x)^2 \)

\[
\int \sin^4 x \, dx = -\sin^3 x \cos x + 3 \int \sin^2 x (1-\sin^2 x) \, dx
\]

Let \( I = \int \sin(x)^4 \, dx \)

\[ I = -\sin(x)^3 \cos(x) + 3 \int \sin(x)^2 \, dx - 3I \]

\[ I = -\frac{1}{4} \sin(x)^3 \cos(x) + \frac{3}{4} \int \sin(x)^2 \, dx \]

Now use solution from previous example.
Ex. 6

Calculate \( \int \tan(x) \, dx \)

Solution:

\[
\int \frac{\sin(x)}{\cos(x)} \, dx = -\int \frac{1}{u} \, du = -\ln|\cos(x)| + C
\]

\( u = \cos(x), \quad du = -\sin(x) \)

Ex. 7

Calculate \( \int \sec(x) \, dx \)

Solution:

\[
\int \sec(x) \, dx = \int \sec(x) \cdot \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)} \, dx
\]

\[
= \int \frac{\sec(x)^2 + \sec(x) \tan(x)}{\tan(x) + \sec(x)} \, dx
\]

\( u = \tan(x) + \sec(x) \)

\( du = (\sec(x)^2 + \sec(x) \tan(x)) \, dx \)
\[
\frac{1}{u} \, du = \ln | \sec(x) + \tan(x) | + C
\]

**Ex. 8**

Calculate \( \int \tan(x)^3 \sqrt{\sec(x)} \, dx \)

**Solution:**

\[
\int \tan(x)^3 \sec(x)^{1/2} \, dx = \int \tan(x)^2 \sec(x)^{-1/2} \sec(x) \tan(x) \, dx
\]

\[
d(\sec(x))
\]

\[
= \int (\sec(x)^2 - 1) \sec(x)^{-1/2} \, dx = \int (u^2 - 1) u^{-1/2} \, du
\]

\[
u = \sec(x), \quad du = \sec(x) \tan(x) \, dx
\]

\[
= \frac{2}{5} u^{5/2} - 2 u^{1/2} + C
\]

\[
= \frac{2}{5} (\sec(x))^{5/2} - 2 (\sec(x))^{1/2} + C
\]
Ex. 9

Calculate \( \int \tan(x)^{2/3} \sec(x)^4 \, dx \).

Solution:

\[
\int \tan(x)^{2/3} \sec(x)^4 \, dx = \int \tan(x)^{2/3} \sec(x)^2 \sec(x)^2 \, dx
\]

\[
d(\tan(x))
\]

\[
= \int \tan(x)^{2/3} (1 + \tan(x)^2) \sec(x)^2 \, dx
\]

\[
u = \tan(x), \quad du = \sec(x)^2 \, dx
\]

\[
= \int u^{2/3} (1 + u^2) \, du = \int (u^{2/3} + u^{8/3}) \, du
\]

\[
= \frac{3}{5} u^{5/3} + \frac{3}{11} u^{11/3} + C
\]

\[
= \frac{3}{5} \tan(x)^{5/3} + \frac{3}{11} \tan(x)^{11/3} + C
\]

Ex. 10

Calculate \( \int \tan(x)^2 \sec(x) \, dx \)
Solution:

\[ \int \tan^2(x) \sec(x) \, dx = \int (\sec^2(x) - 1) \sec(x) \, dx \]

\[ = \int \sec^3(x) \, dx - \int \sec(x) \, dx \]

\[ \ln |\sec(x) + \tan(x)| \]

Now focus on \( \int \sec^3(x) \, dx \).

\[ u = \sec(x) \quad dv = \sec(x)^2 \, dx \]
\[ du = \sec(x) \tan(x) \, dx \quad v = \tan(x) \]

\[ \int \sec^3(x) \, dx = \sec(x) \tan(x) - \int \sec(x) \tan(x)^2 \, dx \]

This integral must contain a factor of \( \tan(x)^2 \), which we rewrite as \( \sec^2(x) - 1 \).

\[ \int \sec^3(x) \, dx = \sec(x) \tan(x) - \int \sec(x) (\sec^2(x) - 1) \, dx \]
Let \( I = \int \sec(x)^3 \, dx \)

\[
I = \sec(x) \tan(x) - I + \int \sec(x) \, dx
\]

\[
2I = \sec(x) \tan(x) + \ln | \sec(x) + \tan(x) |
\]

\[
I = \frac{1}{2} \sec(x) \tan(x) + \frac{1}{2} \ln | \sec(x) + \tan(x) | + C
\]

Putting this altogether, we get

\[
\int \tan(x)^2 \sec(x) \, dx = \frac{1}{2} \sec(x) \tan(x) - \frac{1}{2} \ln | \sec(x) + \tan(x) | + C
\]

**Summary of Strategies**

\[
\int \sin(x)^m \cos(x)^n \, dx
\]

(A) \( m \) odd (\( n \) anything)

- split off factor of \( \sin(x) \)
- rewrite remaining powers of \( \sin(x) \) in terms of \( \cos(x) \) using identity \( \sin(x)^2 = 1 - \cos(x)^2 \)
use the substitution $u = \cos(x)$

(b) $n$ odd (m anything)
- split off factor of $\cos(x)$
- rewrite remaining powers of $\cos(x)$ in terms of $\sin(x)$ using identity $\cos(x)^2 = 1 - \sin(x)^2$
- use the substitution $u = \sin(x)$

c) $m$ and $n$ both even
- rewrite entire integrand in terms of $\sin(x)$ only or $\cos(x)$ only using identity $\cos(x)^2 + \sin(x)^2 = 1$.
- if rewritten in terms of $\sin(x)$...
  - use integration by parts with $dv = \sin(x) \, dx$
  - in resulting integral, rewrite $\cos(x)^2$ as $1 - \sin(x)^2$
  - algebraically solve for original integral.
If rewritten in terms of $\cos(x)$...

- use integration by parts with $dv = \cos(x) \, dx$

- in resulting integral, rewrite $\sin(x)^2$ as $1 - \cos(x)^2$

- algebraically solve for original integral.

\[
\int \tan(x)^m \sec(x)^n \, dx
\]

(A) special cases (memorize)

- $\int \tan(x) \, dx = \ln |\sec(x)| + C$

- $\int \sec(x) \, dx = \ln |\sec(x) + \tan(x)| + C$

(B) $m$ odd ($n$ anything)

- split off factor of $\sec(x) \tan(x)$

- rewrite remaining powers of $\tan(x)$ in terms of $\sec(x)$ using identity
\[
\tan(x)^2 = \sec(x)^2 - 1
\]

- use the substitution \( u = \sec(x) \)

(c) \( n \) even and \( n \neq 2 \) (\( m \) anything)

- split off factor of \( \sec(x)^2 \)
- rewrite remaining powers of \( \sec(x) \) in terms of \( \tan(x) \) using identity
  \[
  \sec(x)^2 = \tan(x)^2 + 1
  \]
- use the substitution \( u = \tan(x) \)

(d) \( m \) even and \( n \) odd

- rewrite entire integrand in terms of \( \sec(x) \) only using identity
  \[
  \tan(x)^2 = \sec(x)^2 - 1
  \]
- use integration by parts with
  \[
  dv = \sec(x)^2 \, dx
  \]
- in resulting integral, rewrite \( \tan(x)^2 \) as \( \sec(x)^2 - 1 \)
- algebraically solve for original integral.
\[ \int \cot(x)^m \csc(x)^n \, dx \]

(A) Special cases (memorize)

- \[ \int \cot(x) \, dx = -\ln |\csc(x)| + C \]
- \[ \int \csc(x) \, dx = -\ln |\csc(x) + \cot(x)| + C \]

(B) m odd (n anything)

- Split off factor of \( \csc(x) \cot(x) \)
- Rewrite remaining powers of \( \cot(x) \) in terms of \( \csc(x) \) using identity
  \[ \cot(x)^2 = \csc(x)^2 - 1 \]
- Use the substitution \( u = \csc(x) \)

(C) n even and \( n \geq 2 \) (m anything)

- Split off factor of \( \csc(x)^2 \)
- Rewrite remaining powers of \( \csc(x) \) in terms of \( \cot(x) \) using identity
  \[ \csc(x)^2 = \cot(x)^2 + 1 \]
• use the substitution \( u = \cot(x) \)

\((D) \) m even and \( n \) odd

• rewrite entire integrand in terms of \( \csc(x) \) only using identity \( \cot(x)^2 = \csc(x)^2 - 1 \)

• use integration by parts with \( dv = \csc(x)^2 \, dx \)

• in resulting integral, rewrite \( \cot(x)^2 \) as \( \csc(x)^2 - 1 \)

• algebraically solve for original integral.
Section 7.3: Trigonometric Substitution

Used to integrate algebraic expressions of one of the following forms:

\[ \sqrt{a^2-x^2}, \quad \sqrt{a^2+x^2}, \quad \sqrt{x^2-a^2} \]

\begin{align*}
\int \frac{1}{\sqrt{1-x^2}} \, dx = \int \frac{1}{\sqrt{1-sin^2(\theta)}} \, cos(\theta) \, d\theta \\
= \int \frac{cos(\theta)}{cos(\theta)} \, d\theta = \int d\theta = \theta + C
\end{align*}
\[ = \sin^{-1}(x) + C \]

**Ex. 2**

Calculate \( \int \sqrt{1-x^2} \, dx \).

**Solution:**

We will use the substitution

\[
\begin{align*}
    x &= \sin(\theta) \\
    dx &= \cos(\theta) \, d\theta
\end{align*}
\]

\[
\sqrt{1-x^2} = \cos(\theta)
\]

\[
\int \sqrt{1-x^2} \, dx = \int \cos(\theta) \cos(\theta) \, d\theta = \int \cos^2(\theta) \, d\theta
\]

Now use integration by parts.

\[
\begin{align*}
    u &= \cos(\theta) & dv &= \cos(\theta) \, d\theta \\
    du &= -\sin(\theta) \, d\theta & v &= \sin(\theta)
\end{align*}
\]

\[
\int \cos^2(\theta) \, d\theta = \cos(\theta) \sin(\theta) + \int \sin^2(\theta) \, d\theta
\]

\[
= \cos(\theta) \sin(\theta) + \int (1-\cos(\theta)^2) \, d\theta
\]
\[
= \cos(\theta)\sin(\theta) + \int \theta \, d\theta - \int \cos(\theta)^2 \, d\theta
\]

So we end up with

\[
\int \cos(\theta)^2 \, d\theta = \frac{1}{2} \cos(\theta)\sin(\theta) + \frac{1}{2} \theta + C
\]

Now rewrite in terms of \( x \).

\[
x = \sin(\theta)
\]

\[
\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \sqrt{1-x^2}
\]

So our final answer is

\[
\int \sqrt{1-x^2} \, dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2} \sin^{-1}(x) + C
\]

**Ex. 3**

Calculate \( \int \frac{1}{x^2\sqrt{2-x^2}} \, dx \).

**Solution:**

We will use the substitution
\[
x = \sqrt{2} \sin(\theta)
\]
\[
dx = \sqrt{2} \cos(\theta) d\theta
\]
\[
\sqrt{2-x^2} = \sqrt{2-2\sin^2(\theta)}
\]
\[
= \sqrt{2} \cos(\theta)
\]
\[
\int \frac{1}{x^2 \sqrt{2-x^2}} \, dx = \int \frac{\sqrt{2} \cos(\theta)}{2 \sin^2(\theta) \sqrt{2} \cos(\theta)} \, d\theta
\]
\[
= \int \frac{1}{2} \csc^2(\theta) \, d\theta = -\frac{1}{2} \cot(\theta) + C
\]

Now rewrite in terms of \( x \):
\[
-\frac{1}{2} \cot\left(\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right)
\]

Not okay! We should simplify.

\[
x = \sqrt{2} \sin(\theta)
\]
\[
\sin(\theta) = \frac{x}{\sqrt{2}} = \frac{\text{opp.}}{\text{hyp.}}
\]
\[
\cot(\theta) = \frac{\text{adj.}}{\text{opp.}} = \frac{\sqrt{2-x^2}}{x}
\]
So our final answer is:

\[ \int \frac{1}{x^2 \sqrt{2-x^2}} \, dx = -\frac{1}{2} \cdot \frac{\sqrt{2-x^2}}{x} + C \]

\[ = \cot(\theta) \]

if \( \sin(\theta) = \frac{x}{\sqrt{2}} \)

---

**Ex. 4**

Calculate \( \int \frac{1}{\sqrt{2x^2 + 1}} \, dx \).

**Solution:**

First some algebra.

\[ \int \frac{1}{\sqrt{2x^2 + 1}} \, dx = \int \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{x^2 + \frac{1}{2}}} \, dx \]

We will use the substitution

\[ x = \frac{1}{\sqrt{2}} \tan(\theta) \]

\[ dx = \frac{1}{\sqrt{2}} \sec(\theta)^2 \, d\theta \]
\[
\sqrt{x^2 + \frac{1}{2}} = \sqrt{\frac{1}{2} \tan(\theta)^2 + \frac{1}{2}} = \frac{1}{\sqrt{2}} \sqrt{\tan(\theta)^2 + 1} = \frac{1}{\sqrt{2}} \sec(\theta)
\]

\[
\int \frac{1}{\sqrt{2}} \frac{1}{\sqrt{x^2 + \frac{1}{2}}} \, dx = \int \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \sec(\theta) \, \frac{1}{\sqrt{2}} \sec(\theta)^2 \, d\theta
\]

\[
= \int \frac{1}{\sqrt{2}} \sec(\theta) \, d\theta = \frac{1}{\sqrt{2}} \ln |\sec(\theta) + \tan(\theta)| + C
\]

Now rewrite in terms of \(x\):

\[
\begin{align*}
\sqrt{2}x^2 + 1 &= \sqrt{2}x \\
\text{hyp} &= \sqrt{2}x \\
\text{adj} &= 1 \\
\sec(\theta) &= \frac{\text{hyp}}{\text{adj}} = \sqrt{2}x \\
\tan(\theta) &= \frac{1}{\sqrt{2}} \tan(\theta) \\
\end{align*}
\]

So our final answer is...

\[
\int \frac{1}{\sqrt{2}x^2 + 1} \, dx = \frac{1}{\sqrt{2}} \ln \left| \sqrt{2}x^2 + 1 + \sqrt{2}x \right| + C
\]
Ex. 5

Calculate \[ \int \frac{1}{(x^2 + q)^{3/2}} \, dx \]

Solution:

We will use the substitution

\[
x = 3 \tan(\theta)
\]

\[
dx = 3 \sec(\theta)^2 \, d\theta
\]

\[
\sqrt{x^2 + q} = 3 \sec(\theta)
\]

\[
\int \frac{1}{(x^2 + q)^{3/2}} \, dx = \int \frac{1}{(3\sec(\theta))^3} \cdot 3 \cdot \sec(\theta)^2 \, d\theta
\]

\[
= \int \frac{3 \sec(\theta)^2}{27 \sec(\theta)^3} \, d\theta = \int \frac{1}{9} \cos(\theta) \, d\theta = \frac{1}{9} \sin(\theta) + C
\]

Now rewrite in terms of \( x \):

\[
\tan(\theta) = \frac{x}{3} = \frac{\text{opp}}{\text{adj}}
\]

\[
\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{q + x^2}}
\]
So our final answer is:

\[ \int \frac{1}{(x^2 + q)^{3/2}} \, dx = \frac{1}{q} \cdot \frac{x}{\sqrt{q + x^2}} + C \]

**Ex. 6**

Calculate \( \int \frac{x^2}{\sqrt{4 + x^2}} \, dx \)

**Solution:**

We will use the substitution

\[ x = 2 \tan(\theta) \]

\[ dx = 2 \sec^2(\theta) \, d\theta \]

\[ \sqrt{4 + x^2} = \sqrt{4 + 4 \tan^2(\theta)} \]

\[ = 2 \sqrt{1 + \tan^2(\theta)} \]

\[ = 2 \sec(\theta) \]

\[ \int \frac{x^2}{\sqrt{4 + x^2}} \, dx = \int \frac{4 \tan^2(\theta)}{2 \sec(\theta)} \cdot 2 \sec(\theta)^2 \, d\theta \]

\[ = \int 4 \tan^2(\theta) \sec(\theta) \, d\theta \]
\[
\int 4 \left( \sec^2(\theta) - 1 \right) \sec(\theta) \, d\theta
\]
\[
= \int 4 \sec^3(\theta) \, d\theta - \int 4 \sec(\theta) \, d\theta
\]

Integration by parts:

Let's do just \( \int \sec^3(\theta) \, d\theta \). Use integration by parts.

\[
\begin{align*}
u &= \sec(\theta) & dv &= \sec^2(\theta) \, d\theta \\
u' &= \sec(\theta) \tan(\theta) & v &= \tan(\theta)
\end{align*}
\]

\[
\int \sec^3(\theta) \, d\theta = \sec(\theta) \tan(\theta) - \int \sec(\theta) \tan^2(\theta) \, d\theta
\]

\[
\tan^2(\theta) = \sec^2(\theta) - 1
\]

\[
\int \sec^2(\theta) \, d\theta = \sec(\theta) \tan(\theta) - \int \sec^3(\theta) \, d\theta + \int \sec(\theta) \, d\theta
\]

\[
\int \sec(\theta)^3 \, d\theta = \frac{1}{2} \sec(\theta) \tan(\theta) + \frac{1}{2} \left( n \int \sec(\theta) + \tan(\theta) \right) + C
\]
Putting it altogether gives:

\[ \int \frac{x^2}{\sqrt{4 + x^2}} \, dx = 2\sec \theta \tan \theta - 2 \ln |\sec \theta + \tan \theta| + C \]

Now rewrite in terms of \( x \):

\[
\begin{align*}
\sqrt{x^2 + 4} & \quad \tan(\theta) = \frac{x}{2} \\
\theta & \quad \sec(\theta) = \frac{\sqrt{x^2 + 4}}{2}
\end{align*}
\]

So our final answer is:

\[ \int \frac{x^2}{\sqrt{4 + x^2}} \, dx = 2 \cdot \frac{\sqrt{x^2 + 4}}{2} \cdot \frac{x}{2} - 2 \ln \left| \frac{\sqrt{x^2 + 4}}{2} + \frac{x}{2} \right| + C \]

Ex. 7

Calculate \( \int \frac{1}{x^2 \sqrt{4x^2 - 36}} \, dx \)

Solution:

First we do some algebra:
\[ \int \frac{1}{x^2 \sqrt{4x^2 - 36}} \, dx = \int \frac{1}{2x^2 \sqrt{x^2 - q}} \, dx \]

We will use the substitution
\[ x = 3 \sec(\theta) \]
\[ dx = 3 \sec(\theta) \tan(\theta) \]
\[ \sqrt{x^2 - q} = 3 \tan(\theta) \]
\[ \Rightarrow \sec(\theta)^2 - 1 = \tan(\theta)^2 \]

\[ \int \frac{1}{2x^2 \sqrt{x^2 - q}} \, dx = \int \frac{3 \sec(\theta) \tan(\theta)}{2 \cdot 9 \sec(\theta)^2 \cdot 3 \tan(\theta)} \, d\theta \]
\[ = \int \frac{1}{18} \cos(\theta) \, d\theta = \frac{1}{18} \sin(\theta) + C \]

Now rewrite in terms of \( x \):

\[ x = 3 \sec(\theta) \]
\[ \sec(\theta) = \frac{x}{3} = \frac{\text{hyp}}{\text{adj}} \]
\[ \sin(\theta) = \frac{\sqrt{x^2 - q}}{x} \]
So our final answer is:
\[
\int \frac{1}{x^2 \sqrt{4x^2 - 36}} \, dx = \frac{1}{18} \cdot \frac{\sqrt{x^2 - 9}}{x} + C
\]

**Ex. 8**

Calculate \( \int \frac{1}{\sqrt{2 + x - x^2}} \, dx \).

**Solution:**
First complete the square:
\[
x^2 - x - 2 = x^2 - x + \frac{1}{4} - 2 - \frac{1}{4}
\]
\[
= (x - \frac{1}{2})^2 - \frac{9}{4}
\]

So our integral is:
\[
\int \frac{1}{\sqrt{\frac{9}{4} - (x - \frac{1}{2})^2}} \, dx
\]

We will use the substitution
\[ x - \frac{1}{2} = \frac{3}{2} \sin(\theta) \]
\[ \frac{dx}{d\theta} = \frac{3}{2} \cos(\theta) \]
\[ \sqrt{\frac{9}{4} - (x - \frac{1}{2})^2} = \sqrt{\frac{9}{4} - \frac{9}{4} \sin^2(\theta)} \]
\[ = \frac{3}{2} \sqrt{1 - \sin^2(\theta)} \]
\[ = \frac{3}{2} \cos(\theta) \]

\[ \int \frac{1}{\sqrt{\frac{9}{4} - (x - \frac{1}{2})^2}} \, dx = \int \frac{1}{\frac{3}{2} \cos(\theta)} \cdot \frac{3}{2} \cos(\theta) \, d\theta \]
\[ = \int d\theta = \theta + C = \sin^{-1}\left(\frac{x - \frac{1}{2}}{3/2}\right) + C \]

**Ex. 9**

**Calculate** \[ \int \frac{1}{(x^2 + 2x + 6)^2} \, dx \]

**Solution:**
First complete the square:
\[
x^2 + 2x + 6 = x^2 + 2x + 1 + 6 - 1 = (x + 1)^2 + 5
\]

So our integral is:
\[
\int \frac{1}{(x+1)^2 + 5} \, dx
\]

We will use the substitution
\[
x + 1 = \sqrt{5} \tan(\theta)
\]
\[
dx = \sqrt{5} \sec(\theta)^2 \, d\theta
\]
\[
\sqrt{(x+1)^2 + 5} = \sqrt{5} \sec(\theta)
\]

\[
\int \frac{1}{(x+1)^2 + 5} \, dx = \int \frac{\sqrt{5} \sec(\theta)^2}{(\sqrt{5} \sec(\theta))^2} \, d\theta
\]
\[
= \frac{\sqrt{5}}{25} \int \cos(\theta)^2 \, d\theta \quad \text{double angle or integration by parts}
\]
\[ \int (\frac{1}{2} + \frac{1}{2} \cos (2\theta)) \, d\theta \]

\[ = \frac{\sqrt{5}}{25} \left( \frac{1}{2} \theta + \frac{1}{4} \sin (2\theta) \right) + C \]

in terms of \( x \)?

\[ = \frac{\sqrt{5}}{25} \left( \frac{1}{2} \theta + \frac{1}{4} \cdot 2 \sin(\theta) \cos(\theta) \right) + C \]

\[ = \frac{\sqrt{5}}{50} \left( \theta + \sin(\theta) \cos(\theta) \right) + C \]

Now rewrite in terms of \( x \):

\[ \tan(\theta) = \frac{x + 1}{\sqrt{5}} \]

\[ \sin(\theta) = \frac{x + 1}{\sqrt{(x+1)^2 + 5}} \]

\[ \cos(\theta) = \frac{\sqrt{5}}{\sqrt{(x+1)^2 + 5}} \]

So our final answer is:
\[
\int \frac{1}{(x^2 + 2x + 6)^2} \, dx = \frac{\sqrt{5}}{50} \left( \tan^{-1} \left( \frac{x+1}{\sqrt{5}} \right) + \frac{\sqrt{5}(x+1)}{(x+1)^2 + 5} \right) + C
\]

**Summary of Strategies**

- Use trigonometric substitution for integrands with quadratic expressions under some integer power of a square root.
- Complete the square as necessary to obtain one of the following forms:
  \[
  \begin{align*}
  \sqrt{a^2 - x^2} \\
  \sqrt{a^2 + x^2} \\
  \sqrt{x^2 - a^2}
  \end{align*}
  \]
  \{ always assume \(a > 0\). \}
- Use the table below as necessary:
Expression | Substitution | Identities

$\sqrt{a^2 - x^2}$

\[\begin{align*}
    x &= a \sin(\Theta) \\
    -\frac{\pi}{2} &\leq \Theta \leq \frac{\pi}{2}
\end{align*}\]

$dx = a \cos(\Theta) \, d\Theta$

$\sqrt{a^2 - x^2} = a \cos(\Theta)$

$\sqrt{a^2 + x^2}$

\[\begin{align*}
    x &= a \tan(\Theta) \\
    -\frac{\pi}{2} &< \Theta < \frac{\pi}{2}
\end{align*}\]

$dx = a \sec^2(\Theta) \, d\Theta$

$\sqrt{a^2 + x^2} = a \sec(\Theta)$

$\sqrt{x^2 - a^2}$

\[\begin{align*}
    x &= a \sec(\Theta) \\
    0 &\leq \Theta < \frac{\pi}{2} \text{ or} \\
    \pi &\leq \Theta < \frac{3\pi}{2}
\end{align*}\]

$dx = a \sec(\Theta) \tan(\Theta) \, d\Theta$

$\sqrt{x^2 - a^2} = a \tan(\Theta)$
Section 7.4: Integrals with Hyperbolic Functions

*We will revisit each integration method but with hyperbolic functions

Note: Be sure to go over the "Hyperbolic Functions Review Sheet".

(A) Substitution

(B) Integration by Parts

→ When choosing $u$, treat hyperbolic and inverse hyperbolic functions as you would treat trigonometric and inverse trigonometric functions

Priority for $u$

L: logarithms

I: inverse trigonometric
  inverse hyperbolic

A: algebraic functions

T: trigonometric
  hyperbolic

E: exponentials
(C) Hyperbolic Integrals

\[ \text{Treat powers of hyperbolic functions as you would treat trigonometric functions. The strategies are identical.} \]

(D) Hyperbolic Substitution

\[ \text{Similar to trigonometric substitution} \]

<table>
<thead>
<tr>
<th>Expression</th>
<th>Trig. Substitution</th>
<th>Hyp. Substitution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{a^2-x^2} )</td>
<td>( x = \sin \theta )</td>
<td>( x = \tanh u )</td>
</tr>
<tr>
<td>( \sqrt{a^2+x^2} )</td>
<td>( x = \tan \theta )</td>
<td>( x = \sinh u )</td>
</tr>
<tr>
<td>( \sqrt{x^2-a^2} )</td>
<td>( x = \sec \theta )</td>
<td>( x = \cosh u )</td>
</tr>
</tbody>
</table>

[Ex. 1]

Write \( \sinh^{-1}(x) \) in terms of logs.

**Solution:**

Suppose \( y = \sinh^{-1}(x) \).

\[ \sinh(y) = x \]

\[ e^y - e^{-y} = x \]
\( e^y - e^{-y} = 2x \)
\( e^{2y} - 1 = 2xe^y \)
\( e^{2y} - 2xe^y - 1 = 0 \)

\[(e^y)^2 - 2x(e^y)' - 1 = 0\]

\( \text{This is a quadratic equation in } e^y \)
\( e^y = 2x \pm \sqrt{(2x)^2 - 4(1)(-1)} \)
\( e^y = x \pm \sqrt{x^2 + 1} \)

Positive \( \sqrt{x^2} - \sqrt{x^2 + 1} \)

So negative root gives negative number.

\( e^y = x + \sqrt{x^2 + 1} \)

\( y = \ln (x + \sqrt{x^2 + 1}) \)
\( y = \text{Shn}^{-1}(x) \)

* See reference sheet for similar formulas for other inverse functions.
Ex. 2

Calculate \( \int x^2 \cosh(x) \, dx \)

\[
\begin{align*}
\text{Solution:} & \quad (LIATE) \quad \text{We will use tabular integration-by-parts.} \\
& \quad \begin{array}{c}
\begin{array}{c}
\text{x}^2 \\
2x \\
2 \\
0
\end{array}
\end{array} \quad \begin{array}{c}
\cosh(x) \\
\sinh(x) \\
\cosh(x) \\
\sinh(x)
\end{array}
\end{align*}
\]

\[
\int x^2 \cosh(x) \, dx = x^2 \sinh(x) - 2x \cosh(x) + 2 \sinh(x) + C
\]

Ex. 2

Calculate \( \int \cosh(x)^2 \, dx \).

\[
\text{Solution:}
\]
Since exponents on both $\cosh(x)$ and $\sinh(x)$ are even, we use integration by parts.

\[
\begin{align*}
    u &= \cosh(x) & du &= \cosh(x) \, dx \\
    du &= \sinh(x) \, dx & v &= \sinh(x)
\end{align*}
\]

\[
\int \cosh(x)^2 \, dx = \cosh(x) \sinh(x) - \int \sinh(x)^2 \, dx
\]

\[
\cosh(x)^2 - \sinh(x)^2 = 1
\]

\[
\sinh(x)^2 = \cosh(x)^2 - 1
\]

\[
\int \cosh^2 x \, dx = \cosh x \sinh x - \int \cosh^2 x \, dx + \int 1 \, dx
\]

\[
= I
\]

\[
I = \int \cosh^2 x \, dx = \frac{1}{2} \cosh(x) \sinh(x) + \frac{x}{2} + C
\]

Alternatively, use the "double-angle" formulas:

\[
\cos(\theta)^2 = \frac{1}{2} + \frac{1}{2} \cos(2\theta)
\]
\[
\cosh(\theta)^2 = \frac{1}{2} + \frac{1}{2} \cosh(2\theta)
\]
\[
\sin(\theta)^2 = \frac{1}{2} - \frac{1}{2} \cos(2\theta)
\]
\[
\Rightarrow -\sinh(\theta)^2 = \frac{1}{2} - \frac{1}{2} \cosh(2\theta)
\]
\[
\sinh(\theta)^2 = -\frac{1}{2} + \frac{1}{2} \cosh(2\theta)
\]

**Ex. 3**

Calculate \( \int \sinh(x)^4 \cosh(x)^3 \, dx \).

**Solution:**

\[
\int \sinh(x)^4 \cosh(x)^3 \, dx =
\]
\[
= \int \sinh(x)^4 \cosh(x)^2 \cosh(x) \, dx
\]
\[
\quad \text{u} = \sinh(x) \quad \Rightarrow \quad du = \cosh(x) \, dx
\]
\[
= \int \sinh(x)^4 (1 + \sinh(x)^2) \cosh(x) \, dx
\]
\[
= \int u^4 (1 + u^2) \, du = \int (u^4 + u^6) \, du
\]
\[
= \frac{1}{5} \sinh(x)^5 + \frac{1}{7} \sinh(x)^7 + C
\]
Calculate $\int \sqrt{x^2 + 16} \, dx$.

(a) trigonometric substitution
(b) hyperbolic substitution

Solution:

(a) We substitute

$$x = 4 \tan(\theta)$$
$$dx = 4 \sec^2(\theta) \, d\theta$$

$$\sqrt{x^2 + 16} = 4 \sec(\theta)$$

$$\int \sqrt{x^2 + 16} \, dx = \int 4 \sec(\theta) \cdot 4 \sec(\theta)^2 \, d\theta$$

$$= 16 \int \sec(\theta)^3 \, d\theta$$

Now compute $\int \sec(\theta)^3 \, d\theta$. We use integration by parts.

$$u = \sec(\theta) \quad dv = \sec(\theta)^2 \, d\theta$$
$$du = \sec(\theta) \tan(\theta) \, d\theta \quad v = \tan(\theta)$$
\[
\int \sec^3(\theta) \, d\theta = \sec(\theta) \tan(\theta) - \int \sec(\theta) \tan(\theta)^2 \, d\theta \\
= \sec(\theta)^2 - 1
\]

\[
\int \sec^3 \theta \, d\theta = \sec(\theta) \tan(\theta) - \int \sec^3 \theta \, d\theta + \int \sec \theta \, d\theta
\]

\[
\int \sec^3 \theta \, d\theta = \frac{1}{2} \sec(\theta) \tan(\theta) + \frac{1}{2} \ln|\sec(\theta) + \tan(\theta)| + C
\]

Now rewrite everything in terms of \(x\).

\[
\sqrt{x^2 + 16}
\]

\[
\tan(\theta) = \frac{x}{4} \quad \sec(\theta) = \frac{\sqrt{x^2 + 16}}{4}
\]

So our final answer is:

\[
\int \sqrt{x^2 + 16} \, dx = 8 \cdot \frac{\sqrt{x^2 + 16}}{4} \cdot \frac{x}{4} + 8 \ln \left| \frac{\sqrt{x^2 + 16}}{4} + \frac{x}{4} \right| + C
\]

(b) We will substitute

\[
x = 4 \sinh(u) \quad \text{dx} = 4 \cosh(u) \, du
\]
\[
\sqrt{x^2 + 16} = \sqrt{16 \sinh(u)^2 + 16} \\
= 4 \sqrt{\sinh(u)^2 + 1} \\
= 4 \sqrt{\cosh(u)^2} \\
= 4 \cosh(u)
\]

\[
\int \sqrt{x^2 + 16} \, dx = \int 4 \cosh(u) \cdot 4 \cosh(u) \, du \\
= 16 \int \cosh(u)^2 \, du \quad \text{(Now use Example 2)}
\]

\[
I = \int \cosh^2 x \, dx = \frac{1}{2} \cosh(x) \sinh(x) + \frac{x}{2} + C
\]

\[
= 8 \cosh(u) \sinh(u) + 8u + C
\]

Now we write in terms of \( x \):

\[
\begin{align*}
\sinh(u) &= \frac{x}{4} \\
\cosh(u) &= \frac{\sqrt{16 + x^2}}{4}
\end{align*}
\]

\[
(ADJ)^2 - (OPP)^2 = (HYP)^2 \\
(ADJ)^2 = 16 + x^2
\]
So our final answer is:

\[ \int \sqrt{16 + x^2} \, dx = 8 \cdot \frac{\sqrt{16 + x^2} \cdot x}{4} + 8 \sinh^{-1} \left( \frac{x}{4} \right) + C \]

---

Ex. 5

Calculate \( \int_{1}^{2} \frac{\sqrt{x^2 - 1}}{x^2} \, dx \).

(Use hyperbolic substitution)

**Solution:**

We will substitute

\[ x = \cosh(u) \]

\[ dx = \sinh(u) \, du \]

\[ \sqrt{x^2 - 1} = \sqrt{\cosh(u)^2 - 1} \]

\[ = \sqrt{\sinh(u)^2} \]

\[ = |\sinh(u)| \]

\[ = \sinh(u) \quad (u > 0) \]

**Limits of Integration:**

\[ x = \cosh(u) \]

\[ x = 1 \quad \Rightarrow \quad u = \cosh^{-1}(1) = 0 \]
\[ x = \sqrt{26} \implies u = \cosh^{-1}(\sqrt{26}) : = a \]

\[
\int_{1}^{\sqrt{26}} \frac{\sqrt{x^2 - 1}}{x^2} \, dx = \int_{0}^{a} \frac{\sinh(u)}{\cosh(u)^2} \, \sinh(u) \, du
\]

\[
= \int_{0}^{a} \tanh(u)^2 \, du = \int_{0}^{a} (1 - \text{sech}(u)^2) \, du
\]

\[
\cosh(u)^2 - \sinh(u)^2 = 1
\]

\[
1 - \tanh(u)^2 = \text{sech}(u)^2
\]

\[
1 - \text{sech}(u)^2 = \tanh(u)^2
\]

\[
= \left( u - \tanh(u) \right) \bigg|_{0}^{a}
\]

\[
= (a - \tanh(a)) - (0 - \tanh(0))
\]

\[
= a - \tanh(a)
\]

Now write as an exact answer.

\[
a = \cosh^{-1}(\sqrt{26})
\]

\[
\cosh(a) = \sqrt{26}
\]

\[
\tanh(a) = \frac{5}{\sqrt{26}}
\]
\[(ADT)^2 - (OPP)^2 = (HYP)^2\]
\[(\sqrt{26})^2 - (5)^2 = 1\]

So our final answer is:
\[
\int_1^{\sqrt{26}} \frac{\sqrt{x^2 - 1}}{x^2} \, dx = \cosh^{-1}(\sqrt{26}) - \frac{5}{\sqrt{26}}
\]

**Ex. 6**

Calculate
\[
\int \frac{\sqrt{9 + x^2}}{x} \, dx
\]

**Solution:**

We can substitute

- **Trigonometric:** \( x = 3 \tan(\theta) \)
- **Hyperbolic:** \( x = 3 \sinh(u) \)

\[dx = 3 \cosh(u) \, du\]
\[\sqrt{9 + x^2} = 3 \cosh(u)\]
\[(1 + \sinh(u)^2 = \cosh(u)^2)\]

\[
\int \frac{\sqrt{9 + x^2}}{x} \, dx = \int \frac{3 \cosh(u)}{3 \sinh(u)} \cdot 3 \cosh(u) \, du
\]
\[
= 3 \int \frac{\cosh(u)^2}{\sinh(u)} \, du \quad \left( \int \cos(x)^2 \sin(x)^{-1} \, dx \right)
\]

\[
= 3 \int \frac{1 + \sinh(u)^2}{\sinh(u)} \, du
\]

\[
= 3 \int (\text{csch}(u) + \sinh(u)) \, du
\]

\[
= 3 \int \text{csch}(u) \, du + 3 \int \sinh(u) \, du
\]

\[
= -\ln|\text{csch}(u) + \coth(u)| = \cosh(u)
\]

\[
= -3 \ln|\text{csch}(u) + \coth(u)| + 3 \cosh(u) + C
\]

Now rewrite in terms of \( x \):

\[
\text{csch}(u) = \frac{3}{x}
\]

\[
\coth(u) = \frac{\sqrt{x^2 + 9}}{x}
\]

\[
\cosh(u) = \frac{\sqrt{x^2 + 9}}{3}
\]

\[
\sinh(u) = \frac{x}{3}
\]
So our final answer is:

\[ \int \frac{\sqrt{9 + x^2}}{x} \, dx = -3 \ln \left| \frac{3}{x} + \frac{\sqrt{9 + x^2}}{x} \right| + \sqrt{9 + x^2} + C \]
Section 7.5: Method of Partial Fractions

**Def:** If the polynomial $p(x)$ can be written as

$$p(x) = q(x) \cdot r(x)$$

where $q$ and $r$ are non-constant polynomials, we say $p$ is reducible. Otherwise, $p$ is irreducible.

**Thm:** (Fundamental Theorem of Algebra) If $p(z)$ is a polynomial, then $p(z)$ has a root in $C$.

This means every polynomial over $C$ can be factored into linear factors.

$$p(z) = A(z-z_1)^{m_1}(z-z_2)^{m_2} \cdots (z-z_k)^{m_k}$$

**Thm:** If $p(x)$ is an irreducible polynomial over $\mathbb{R}$, then $p$ has
one of the following two forms:

1. \( ax + b \)
2. \( ax^2 + bx + c \) with \( b^2 - 4ac < 0 \)

**Ex:**

1. \( x^2 + 1 \) is irreducible over \( \mathbb{R} \)
2. \( x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1) \)

Every rational function can be decomposed into partial fractions.

\[
\frac{P(x)}{Q(x)} = r(x) + \sum_{j} \frac{P_j(x)}{Q_j(x)}
\]

1. \( \deg (p_j) < \deg (Q_j) \)
2. \( Q_j \) is irreducible
3. \( Q_1 Q_2 \cdots Q_N = Q \)

This is like "un-simplifying".
**Ex:** \[
\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{1/2}{x-1} + \frac{1/2}{x+1}
\]

**Ex. 1**

Calculate \( \int \frac{1}{x^2 - 7x + 10} \, dx \).

**Solution:**

1. Factor denominator:
   \[x^2 - 7x + 10 = (x-5)(x-2)\]
   * distinct, linear factors

2. So that means the partial fraction decomposition (PFD) has the form:
   \[
   \frac{1}{x^2-7x+10} = \frac{A}{x-5} + \frac{B}{x-2}
   \]
   * This equation holds for all \( x \).

Now solve for \( A \) and \( B \).

(multiply both sides by \( x^2 - 7x + 10 \))
\[ I = A(x-2) + B(x-5) \]

**Method 1**

\[ 0x + 1 = (A+B)x + (-2A - 5B) \]

Now equate like coefficients:

- \[ A + B = 0 \] (\( x^1 \) - coefficients)
- \[ -2A - 5B = 1 \] (\( x^0 \) - coefficients)

\[
\begin{align*}
2A + 2B &= 0 \\
-2A - 5B &= 1 \\
0 - 3B &= 1 \\
B &= -\frac{1}{3}, \quad A = \frac{1}{3}
\end{align*}
\]

**Method 2**

Our master equation is true for all values of \( x \). So choose clever values of \( x \).

**Master:** \[ I = A(x-2) + B(x-5) \]

\[ x = 2: \quad 1 = 0 + B(-3) \implies B = -\frac{1}{3} \]
\[
x = 5: \quad 1 = A(3) + 0 \quad \Rightarrow \quad A = \frac{1}{3}
\]

So our PFD is

\[
\frac{1}{x^2 - 7x + 10} = \frac{1/3}{x - 5} + \frac{-1/3}{x - 2}
\]

So our integral is...

\[
\int \frac{1}{x^2 - 7x + 10} \, dx = \int \left(\frac{1/3}{x - 5} - \frac{1/3}{x - 2}\right) \, dx
\]

\[
= \frac{1}{3} \ln |x - 5| - \frac{1}{3} \ln |x - 2| + C
\]

\[
\textbf{Ex. 2}
\]

Calculate \( \int \frac{x^2 - 2}{(x-1)(2x+3)(x+1)} \, dx \).

\underline{Solution:}

(Distinct linear factors)

\[
\frac{x^2 - 2}{(x-1)(2x+3)(x+1)} = \frac{A}{x-1} + \frac{B}{2x+3} + \frac{C}{x+1}
\]

Now use Method #2 to find \( A, B, C \).

\[
x^2 - 2 = A(2x+3)(x+1) + B(x-1)(x+1) + C(x-1)(2x+3)
\]
\[ x = 1 : \quad -1 = A (5)(2) + 0 + 0 \quad \Rightarrow \quad A = -\frac{1}{10} \]

\[ x = -\frac{3}{2} : \quad \frac{1}{4} = 0 + B\left(-\frac{5}{2}\right)\left(-\frac{1}{2}\right) + 0 \quad \Rightarrow \quad B = \frac{1}{5} \]

\[ x = -1 : \quad -1 = 0 + 0 + C (-2)(1) \quad \Rightarrow \quad C = \frac{1}{2} \]

Now the integral:

\[
\int \frac{x^2 - 2}{(x-1)(2x+3)(x+1)} \, dx = \int \left( -\frac{1/10}{x-1} + \frac{1/5}{2x+3} + \frac{1/2}{x+1} \right) \, dx
\]

\[ = -\frac{1}{10} \ln|x-1| + \frac{1}{10} \ln|2x+3| + \frac{1}{2} \ln|x+1| + C' \]

**Ex. 3**

Calculate \[ \int \frac{x^2 + 11x}{(x-1)(x+1)^2} \, dx \]

**Solution:**
In general, this means our PFD is:
\[
\frac{x^2 + 11x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{Bx + C}{(x+1)^2}
\]

This term can be written slightly differently.

There are constants \( B' \) and \( C' \):
\[
\frac{Bx + C}{(x+1)^2} = \frac{B'}{x+1} + \frac{C'}{(x+1)^2}
\]

In general, each repeated factor contributes a PFD of the form
\[
\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \ldots + \frac{A_m}{(x-a)^m}
\]

where \( m = \) degree of repeated factor.

So the PFD for our integrand has the form:
\[
\frac{x^2 + 11x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}
\]
Now determine \( A, B, C \).

**Master equation:**

Multiply by \( (x-1)(x+1)^2 \):

\[
x^2 + 11x = A (x+1)^2 + B (x-1)(x+1) + C (x-1)
\]

\[x = -1: \quad -10 = 0 + 0 + C (-2) \quad \Rightarrow \quad C = 5\]

\[x = 1: \quad 12 = A(4) + 0 + 0 \quad \Rightarrow \quad A = 3\]

\[x = 0: \quad 0 = 3(1) + B(-1)(1) + 5(-1) \quad \Rightarrow \quad B = -2\]

So now our integral is:

\[
\int \frac{x^2 + 11x}{(x-1)(x+1)^2} \, dx = \int \left( \frac{3}{x-1} - \frac{2}{x+1} + \frac{5}{(x+1)^2} \right) \, dx
\]

\[= 3 \ln(x-1) - 2 \ln(x+1) - \frac{5}{x+1} + C\]
What if $\frac{p(x)}{q(x)}$ is not proper?

Long division of polynomials!

Ex. 4

Calculate $\int \frac{x^3+1}{x^2-4} \, dx$

Solution:

Since $\deg$ (numerator) is not less than $\deg$ (denominator), we must do long division first.

\[
x^2 - 4 \overset{x}{\overline{\underline{x^3 + 0x^2 + 0x + 1}}} \]
\[
-x^3 - 4x \quad \frac{4x + 1}{4x + 1}
\]

This means

\[
\frac{x^3+1}{x^2-4} = x + \frac{4x+1}{x^2-4}
\]

Now we find the PFD of the
remainder term.

\[ \frac{4x+1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} \]

Now find A and B.

**Master:** \[ 4x+1 = A(x+2) + B(x-2) \]

\[ x = 2: \quad 9 = A(4) + 0 \]

\[ \Rightarrow A = \frac{9}{4} \]

\[ x = -2: \quad -7 = 0 + B(-4) \]

\[ \Rightarrow B = \frac{7}{4} \]

Our integral is...

\[ \int \frac{x^3+1}{x^2-4} \, dx = \int \left( x + \frac{9/4}{x-2} + \frac{7/4}{x+2} \right) \, dx \]

\[ = \frac{1}{2} x^2 + \frac{9}{4} \ln |x-2| + \frac{7}{4} \ln |x+2| + C \]

**Ex. 5**

Calculate \( \int \frac{18}{(x+3)(x^2+9)} \, dx \).
Solution:  
Our PFD has the form  
\[
\frac{18}{(x+3)(x^2+9)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+9}
\]
Now find \(A, B, C\):  

Master:  
\[
18 = A(x^2+9) + (Bx+C)(x+3)
\]

\(x = -3\):  
\[
18 = A(18) + 0 \quad \Rightarrow \quad A = 1
\]

\(x = 0\):  
\[
18 = B(0) + C(3) \quad \Rightarrow \quad 3C = 18 \quad \Rightarrow \quad C = 6
\]

\(x = 1\):  
\[
18 = (1+B)(1) + (3B+C) + (9+3C) \quad \Rightarrow \quad 18 = (1+B) + (3B+C) + (9+3C)
\]

Problem! Sometimes these equations are not independent. 
So let's expand instead:  
\[
18 = x^2 + 9 + Bx^2 + 3Bx + Cx + 3C
\]

\[
18 = (1+B)x^2 + (3B+C)x + (9+3C)
\]

Now set like coefficients equal.  
\[
G = 1+B \quad \Rightarrow \quad B = -1
\]
\[
0 = 3B + C \quad \Rightarrow \quad C = 3
\]
So now our integral is...

\[ \int \frac{18}{(x+3)(x^2+9)} \, dx = \int \left( \frac{1}{x+3} + \frac{-x+3}{x^2+9} \right) \, dx \]

\[ = \int \left( \frac{1}{x+3} - \frac{x}{x^2+9} + \frac{3}{x^2+9} \right) \, dx \]

\[ u = x^2 + 9 \quad x = 3 \tan(\theta) \]

\[ = \ln |x+3| - \frac{1}{2} \ln (x^2+9) + \tan^{-1} \left( \frac{x}{3} \right) + K \]

**Useful to Memorize**

\[ \int \frac{x}{x^2+a^2} \, dx = \frac{1}{2} \ln (x^2+a^2) + C \]

\[ \int \frac{1}{x^2+a^2} \, dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C \]

**Ex. b**

Calculate \[ \int \frac{25}{x \left( x^2 + 2x + 5 \right)^2} \, dx \]
Solution:

Q: Can this quadratic be factored? 
A: No! \( \Delta = (2)^2 - 4(5)(1) = -16 < 0 \).

The form of the PFD is:

\[
\frac{25}{x(x^2+2x+5)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+5} + \frac{Dx+E}{(x^2+2x+5)^2}
\]

Now find the coefficients.

Master:

\[
25 = A(x^2+2x+5)^2 + (Bx+C)(x^2+2x+5)x + (Dx+E)x
\]

\( x = 0 \) : \( 25 = A(25) + 0 + 0 \)

\( \iff \) \( A = 1 \)

Now expand to find \( B, C, D, E \). We can write master equation as:

(recall \( A = 1 \))

\[
25 = (x^2+2x+5)[x^2+2x+5 + (Bx+C)x] + (Dx+E)x
\]

\[
25 = (x^2+2x+5)((B+1)x^2 + (C+2)x + 5) + Dx^2 + Ex
\]
No need to fully expand. Just look for like coefficients on each side:

<table>
<thead>
<tr>
<th>Power of x</th>
<th>Coefficient on left</th>
<th>Coefficient on right</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^4$</td>
<td>0</td>
<td>$B+1$</td>
</tr>
</tbody>
</table>

Therefore, $0 = B + 1 \implies B = -1$

New master: $25 = (x^2 + 2x + 5)((c+2)x + 5) + Dx^2 + Ex$

<table>
<thead>
<tr>
<th>Power of x</th>
<th>Coefficient on left</th>
<th>Coefficient on right</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^3$</td>
<td>0</td>
<td>$C + 2$</td>
</tr>
</tbody>
</table>

Therefore, $0 = C + 2 \implies C = -2$

New master: $25 = (x^2 + 2x + 5)(5) + Dx^2 + Ex$

$$25 = (D+5)x^2 + (E+10)x + 25$$

<table>
<thead>
<tr>
<th>Power of x</th>
<th>Coefficient on left</th>
<th>Coefficient on right</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2$</td>
<td>0</td>
<td>$D+5$</td>
</tr>
<tr>
<td>$x$</td>
<td>0</td>
<td>$E+10$</td>
</tr>
<tr>
<td>1</td>
<td>25</td>
<td>25</td>
</tr>
</tbody>
</table>

Therefore, $D + 5 = 0 \implies D = -5$

Therefore, $E + 10 = 0 \implies E = -10$

So in summary we have:

- $A = 1$
- $C = -2$
- $E = -10$
- $B = -1$
- $D = -5$
So now our integral is:

\[ \int (\ldots) \, dx = \int \left( \frac{1}{x} - \frac{x + 2}{x^2 + 2x + 5} - \frac{5x + 10}{(x^2 + 2x + 5)^2} \right) \, dx \]

Now do each piece separately:

(A) \[ \int \frac{1}{x} \, dx = \ln |x| \]

(B) Complete the square:

\[ \int \frac{x + 2}{x^2 + 2x + 5} \, dx = \int \frac{x + 2}{(x + 1)^2 + 4} \, dx \]

Now make the substitution

\[ u = x + 1, \quad du = dx \]

\[ = \int \frac{u + 1}{u^2 + 4} \, dx = \int \frac{u}{u^2 + 4} \, dx + \int \frac{1}{u^2 + 4} \, dx \]

\[ v = u^2 + 4, \quad u = 2 \tan(\theta) \] (or memorized formulas)
\[
= \frac{1}{2} \ln (u^2 + 4) + \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right)
\]
\[
= \frac{1}{2} \ln (x^2 + 2x + 5) + \frac{1}{2} \tan^{-1}\left(\frac{x + 1}{2}\right)
\]

\textcircled{C} \text{ Complete the square:}
\[
\int \frac{5x + 10}{(x^2 + 2x + 5)^2} \, dx = \int \frac{5u + 10}{(u^2 + 4)^2} \, dx
\]

\textit{Now make the substitution}
\[
u = x + 1, \quad du = dx
\]
\[
\int \frac{5u + 5}{(u^2 + 4)^2} \, du = \int \frac{5u}{(u^2 + 4)^2} \, du + \int \frac{5}{(u^2 + 4)^2} \, du
\]

\textcircled{D} \text{ Use the substitution}
\[
u = u^2 + 4, \quad \frac{1}{2} \, dv = du
\]
\[
\int \frac{5u}{(u^2 + 4)^2} \, du = \int \frac{5}{2} \cdot \frac{1}{v^2} \, dv = -\frac{5}{2v} = -\frac{5}{2} \cdot \frac{1}{x^2 + 2x + 5}
\]
Use the substitution

\[ u = 2 \tan(\theta), \quad du = 2 \sec(\theta)^2 \, d\theta \]

\[
\int \frac{5}{(u^2+4)^2} \, du = \int \frac{5 - 2 \sec(\theta)^2}{(4 \tan(\theta)^2+4)^2} \, d\theta
\]

\[ = \int \frac{5}{8} \cos(\theta)^2 \, d\theta \quad \text{half-angle formula or integration by parts (see section 7.3 Ex #2)} \]

\[ = \frac{5}{16} \cos(\theta) \sin(\theta) + \frac{5}{16} \theta \]

Rewrite in terms of \( u \):

\[
\sqrt{u^2+4} \quad u \quad 2 \quad \theta
\]

\[
\tan(\theta) = \frac{u}{2} \quad \cos(\theta) = \frac{2}{\sqrt{u^2+4}} \quad \sin(\theta) = \frac{u}{\sqrt{u^2+4}}
\]

\[
\int \frac{5}{(u^2+4)^2} \, du = \frac{5}{16} \cdot \frac{2}{\sqrt{u^2+4}} \cdot \frac{u}{\sqrt{u^2+4}} + \frac{5}{16} \cdot \tan^{-1}\left(\frac{u}{2}\right)
\]

\[ = \frac{5}{8} \cdot \frac{x+1}{x^2+2x+5} + \frac{5}{16} \tan^{-1}\left(\frac{x+1}{2}\right) \]

Now we can combine all of our work
\[
\int \frac{25}{x(x^2+2x+5)^2} \, dx = \boxed{A} - \boxed{B} - \boxed{D} - \boxed{E} - \boxed{C}
\]

\[
= \ln |x| - \frac{1}{2} \ln (x^2 + 2x + 5) - \frac{1}{2} \tan^{-1}\left(\frac{x+1}{2}\right)
\]

\[
+ \frac{5}{2} \cdot \frac{1}{x^2+2x+5} - \frac{5}{8} \cdot \frac{x+1}{x^2+2x+5} - \frac{5}{16} \tan^{-1}\left(\frac{x+1}{2}\right)
\]

(Plus some constant of integration.)

This can be simplified by combining logarithms and like terms.

\[
\int \frac{25}{x(x^2+2x+5)^2} \, dx =
\]

\[
= \ln\left(\frac{1x1}{\sqrt{x^2+2x+5}}\right) - \frac{13}{16} \tan^{-1}\left(\frac{x+1}{2}\right) - \frac{5x - 15}{8(x^2 + 2x + 5)} + K
\]

Phew!
Section 6.2: Average value, Density, Volume

Average Value of a Function

Given $N$ numbers $a_1, \ldots, a_N$, their mean (simple average) is

$$\frac{a_1 + a_2 + \ldots + a_N}{N} = \frac{1}{N} \sum_{k=1}^{N} a_k$$

Average value of a function would have infinitely many $a_k$. But:

$$R_N = \frac{b-a}{N} \left( f(y_1) + f(x_2) + \ldots + f(x_N) \right)$$

Riemann sum with right endpoints and $N$ rectangles

$$\frac{1}{b-a} R_N = \frac{f_1 + f_2 + \ldots + f_N}{N}$$

average of $N$ specific values of $f$

What if $N \to \infty$??

$$\lim_{N \to \infty} \frac{f_1 + \ldots + f_N}{N} = \lim_{N \to \infty} \frac{R_N}{b-a} = \frac{1}{b-a} \int_a^b f(x) \, dx$$
**Def:** The average value of $f(x)$ on $[a, b]$ is the number

$$M = \frac{1}{b-a} \int_a^b f(x) \, dx$$

**Note that**

$$M (b-a) = \int_a^b f(x) \, dx$$

(area of rectangle with base on $[a, b]$ and height $M$)

(area under graph of $y = f(x)$)

**Thm:** If $f(x)$ is continuous on $[a, b]$ then $f$ takes on the value $M$ at least once in $[a, b]$. 
Calculate the average value of
\[ f(x) = a + b \sin (cx) \]
on over one period. \((c > 0)\)

**Solution:**

The period of \( f \) is \( P = \frac{2\pi}{c} \). So an arbitrary interval that covers one period is

\[ I = [x_0, x_0 + C] \]

So the average of \( f \) on \( I \) is

\[
M = \frac{1}{x_0 + \frac{2\pi}{c} - x_0} \int_{x_0}^{x_0 + \frac{2\pi}{c}} \left(a + b \sin (cx)\right) \, dx
\]

\[
M = \frac{c}{2\pi} \left( a \cdot \frac{2\pi}{c} - \frac{b}{c} \cos (cx) \right) \bigg|_{x_0}^{x_0 + \frac{2\pi}{c}}
\]

\[
M = \frac{c}{2\pi} \left[ a \cdot \frac{2\pi}{c} - \frac{b}{c} \left( \cos (cx_0 + 2\pi) - \cos (cx_0) \right) \right]
\]

\[= 0\]
\[ M = \frac{c}{2\pi} \cdot a \cdot \frac{2\pi}{c} = a \]

Does this make sense graphically?

\[ y = a + b \sin(\alpha x) \]

Density (Mass, Charge, Population, etc.)

Consider a thin rod:

\[ \lambda = \lambda(x) = \text{density at a length } x \text{ along the rod} \]

To get total mass of rod, we consider one small piece of the rod

\[ \lambda(x_1) \]

\[ \lambda(x_2) \]

\( x \): location of left endpoint
$x_2$: location of right endpoint
$ar{x}$: location of midpoint

When we say the piece is “small” we mean the following:

1. Length of piece: $x_2 - x_1 = dx$
   $\Rightarrow$ $dx$ is an arbitrarily small number

2. Even though the density $\lambda$ varies across the piece, we approximate $\lambda$ as constant
   $\Rightarrow$ so $\lambda = \lambda(\bar{x})$ for entire piece

So now what is the mass of the piece?

$$\text{mass} = (\text{density}) \cdot (\text{length})$$

$$dm = \lambda(\bar{x}) \cdot dx$$

The total mass of the rod is the “sum” of all the small masses. But the “sum” of all $dm$’s is an integral

```
total mass: $m = \int dm = \int_0^L \lambda(x) \, dx$
```
**Ex. 2**

Find total mass of 2m-rod with 
\[ \lambda(x) = 1 + 0.5 \sin(\pi x) \text{ (kg/m)} \] for 
\[ 0 \leq x \leq 2. \]

**Solution:**
The total mass is 
\[
m = \int_0^2 \lambda(x) \, dx = \int_0^2 (1 + 0.5 \sin(\pi x)) \, dx = 2
\]

For general objects (2D or 3D) the total mass is also an integral:

- **two-dimensional plate**: 
  \[ \iint \rho(x,y) \, dx \, dy \]
- **three-dimensional blob**: 
  \[ \iiint \rho(x,y,z) \, dx \, dy \, dz \]

* These double and triple integrals are discussed in Math 251.

We will discuss one special case of two-dimensional density, an axially symmetric plate.
An axially symmetric plate is a plate whose density depends only on the variable $r$, the distance from the symmetry axis.

The density at all points on the circle is $\rho(r)$. The density depends only on the distance $r$.

We can calculate the total mass of the plate by considering a thin ring (i.e., divide the plate into small pieces).

inner radius: $r_1$
outer radius: $r_2$
average of inner and outer radius: $\bar{r} = \frac{r_1 + r_2}{2}$
When we say the ring is “thin” we mean the following:

1. **Width of ring:** \( r_2 - r_1 = dr \)
   \( \Rightarrow \) \( dr \) is an arbitrarily small number

2. Even though the density \( \sigma \) varies across the ring, we approximate \( \sigma \) as constant
   \( \Rightarrow \) So \( \sigma = \sigma(\bar{r}) \) for entire ring

So now what is the mass of the ring?

\[
\text{mass} = (\text{density}) \cdot (\text{area})
\]

\[
dm = \sigma(\bar{r}) \cdot (\pi r_2^2 - \pi r_1^2)
\]  
(outer disk minus inner disk)

\[
dm = \sigma(\bar{r}) \cdot \pi \left( r_2 + r_1 \right) \left( r_2 - r_1 \right)
\]

\[
dm = \sigma(\bar{r}) \cdot 2\pi \left( \frac{r_2 + r_1}{2} \right) \cdot \left( r_2 - r_1 \right)
\]

\[
= \bar{r} = dr
\]

\[
dm = 2\pi \bar{r} \sigma(\bar{r}) \, dr
\]

The total mass of the plate is the “sum” of all the small masses. But the “sum” of all \( dm \)'s is an integral
total mass: \[ m = \int dm = \int_0^R 2\pi r \sigma(r) \, dr \]

(The factor \(2\pi r\) takes into account that outer rings are larger than inner rings, and so contribute relatively more to the total mass.)

Ex. 3

A circular plate of radius \( R \) has density

\[ \sigma(r) = \frac{\sigma_0 R^2 r}{(R^2 + r^2)^{3/2}} \quad \text{(kg/m}^2\text{)} \]

What is the total mass of the plate?

Solution:

Remark: A graph of \( \sigma(r) \) is below:
So the plate is brittle (low density) near its center, has a maximum density at about \( \frac{1}{\sqrt{2}} \) (\( \approx 70\% \)) of its radius, and the density decreases thereafter.

The total mass is

\[
m = \int_0^R 2\pi \Gamma_0 R^2 \frac{r^2}{(R^2 + r^2)^{3/2}} \, dr
\]

In physics we often are more interested in how variables scale with each other, not necessarily the exact relationship. To illustrate this, we will non-dimensionalize the integral by removing all parameters from the integral. (This is optional but fun!)

\[
m = 2\pi \Gamma_0 R^2 \int_0^R \frac{r^2}{(R^2 (1 + r^2/R^2))^{3/2}} \, dr
\]
\[
m = \frac{2\pi \tau_0}{R} \int_0^R \frac{r^2}{\left(1 + \frac{r^2}{R^2}\right)^{3/2}} \, dr
\]

substitute \( u = \frac{r}{R} \)

\( (Ru = r, \ Rdu = dr) \)

\[
m = \frac{2\pi \tau_0}{R} \int_0^1 \frac{(Ru)^2}{(1 + u^2)^{3/2}} \cdot (Rdu)
\]

\[
m = 2\pi \tau_0 R^2 \int_0^1 \frac{u^2}{(1 + u^2)^{3/2}} \, du
\]

\[
\int_0^1 \frac{u^2}{(1 + u^2)^{3/2}} \, du
\]

Some number \((\approx 0.174)\)

So we can answer questions like:

"If the radius is doubled, by what factor does the mass increase?" (4)

We can use trigonometric or hyperbolic substitution to compute the integral.

\[
\begin{align*}
    u &= \tan(\theta) \\
    du &= \sec^2(\theta) \, d\theta \\
    1 + u^2 &= \sec^2(\theta)
\end{align*}
\]

\[
\begin{align*}
    u &= \sinh(\theta) \\
    du &= \cosh(\theta) \, d\theta \\
    1 + u^2 &= \cosh^2(\theta)
\end{align*}
\]
Which do you want to use? Why not both?

**Trigonometric Substitution**

\[
\int_0^1 \frac{u^2}{(1+u^2)^{3/2}} \, du = \int_0^{\pi/4} \frac{\tan(\theta)^2}{\sec(\theta)^3} \, d\theta
\]

\[
= \int_0^{\pi/4} \frac{\tan(\theta)}{\sec(\theta)} \, d\theta = \int_0^{\pi/4} \frac{\sec(\theta)^2 - 1}{\sec(\theta)} \, d\theta
\]

\[
= \int_0^{\pi/4} (\sec(\theta) - \cos(\theta)) \, d\theta
\]

\[
= \left( \ln |\sec(\theta) + \tan(\theta)| - \sin(\theta) \right) \bigg|_0^{\pi/4}
\]

\[
= \ln \left( \sqrt{2} + 1 \right) - \frac{1}{\sqrt{2}}
\]

**Hyperbolic Substitution**

\[
\int_0^1 \frac{u^2}{(1+u^2)^{3/2}} \, du = \int_0^\alpha \frac{\sinh(\theta)^2}{\cosh(\theta)^3} \cosh(\theta) \, d\theta
\]

\[
(\alpha = \sinh^{-1}(1))
\]

\[
= \int_0^\alpha \tanh(\theta)^2 \, d\theta = \int_0^\alpha (1 - \text{sech}(\theta)^2) \, d\theta
\]
\[
\left(\theta - \tanh(\theta)\right)\bigg|_0^\alpha = \alpha - \tanh(\alpha)
\]

Use a (hyperbolic) triangle to get \( \tanh(\alpha) \).

\[\begin{align*}
\sinh(\alpha) &= 1 \\
\tanh(\alpha) &= \frac{1}{\sqrt{2}}
\end{align*}\]

\[
(\sqrt{2})^2 - 1^2 = 1^2
\]

So the final answer is

\[
\int_0^1 \frac{u^2}{(1 + u^2)^{3/2}} \, du = \sinh^{-1}(1) - \frac{1}{\sqrt{2}}
\]

(Recall: \( \sinh^{-1}(x) = \ln(x + \sqrt{1+x^2}) \), so...

\[
\sinh^{-1}(1) = \ln(1 + \sqrt{2})
\]

**Volumes**

What is a "cylinder"? An object for which cross-sections parallel to a base are all congruent to each other.

(So a cube is a cylinder? A prism? Yes!)
Volume of cylinder: \( V = Ah \)

Now consider some amorphous solid, whose cross-sectional varies with height, but is a known function \( A(y) \).

As before, we find the total volume by dividing the solid into "small" cylindrical slabs.

height of lower base of slab: \( y_1 \)
height of upper base of slab: \( y_2 \)
height of midplane of slab: \( \bar{y} = \frac{y_1 + y_2}{2} \)

When we say the slab is "thin," we mean the following:

1. height of slab: \( y_2 - y_1 = dy \)
   \( \rightarrow \) \( dy \) is an arbitrarily small number

2. Even though the cross-sectional area varies with height, we approximate the cross-sectional area of slab as constant:
   \( \rightarrow \) \( A = A(\bar{y}) \) for entire slab

So now what is the volume of the slab?

\[
\text{volume} = (\text{cross-sectional area}) \cdot (\text{height})
\]
\[
dV = A(\bar{y}) \cdot dy
\]

The total volume of the solid is the "sum of all the small volumes. But the "sum" of all \( dV \)'s is an integral

\[
\text{total volume} : \quad V = \int dV = \int_a^b A(y) \, dy
\]
Find formula for volume of pyramid with square cross-sections parallel to base. The base has length \(L\) and the pyramid has height \(H\).

Solution:
Each cross-section is a square whose area depends on its height \(y\).
Our first goal is to find a formula for \(A(y)\).

Let \(s(y)\) be the side length of any square cross-section.

Side view of cross section:
Similar triangles gives us:
\[
\frac{H}{L} = \frac{H-y}{s(y)} \quad \Rightarrow \quad s(y) = \frac{L}{H} (H-y)
\]
The cross-sectional area is thus
\[
A(y) = s(y)^2 = \frac{L^2}{H^2} (H-y)^2
\]
The total volume is then
\[
V = \int_0^H A(y)\,dy = \int_0^H \frac{L^2}{H^2} (H-y)^2 \,dy
\]
(use \( u = \frac{H-y}{H} \) to non-dimensionalize!)
\[
= \int_1^0 \frac{L^2}{H^2} (Hu)^2 \cdot (-H\,du) = L^2H \int_0^1 u^2\,du
\]
just a number!
(So volume scales as \( L^2H \). So double base length to quadruple volume.) Of course,
\[
V = L^2H \int_0^1 u^2\,du = L^2H \left( \frac{u^3}{3} \right) \bigg|_0^1 = \frac{1}{3} L^2H
\]
This is a formula you likely already knew. But now you know where the factor of \( \frac{1}{3} \) comes from.
Ex. 5

The base of a solid is the disk $x^2 + y^2 \leq 1$. Cross-sections perpendicular to the x-axis are triangles whose base and height have equal length. Find the volume of the solid.

Solution:

The points A and B have coordinates

$A = (x, \sqrt{1-x^2})$

$B = (x, -\sqrt{1-x^2})$

Write in terms of $x$ since slice is vertical.

The dimensions of the cross-section are thus:
The base of a solid is the region bounded by \( y = x^2 \) and \( x = y^2 \). Cross sections perpendicular to the \( y \)-axis are equilateral triangles. Find the volume of the solid.

**Solution:**

What is the area of an equilateral triangle?

- **Diagram:**
  - \( B = \frac{L}{2} \)
  - \( H = L \sin (60^\circ) = \frac{\sqrt{3}}{2} L \)
  - \( L = \) length of one side
Area = 2 \cdot (\frac{1}{2} BH) = BH = \frac{\sqrt{3}}{4} L^2

Now back to our solid.

\( P = (y^2, y) \quad L = |PQ| = \sqrt{y} - y^2 \)

\( Q = (\sqrt{y}, y) \quad A(y) = \frac{\sqrt{3}}{4} (\sqrt{y} - y^2)^2 \)

So the total volume is:

\[ V = \int_0^1 A(y) \, dy = \frac{\sqrt{3}}{4} \int_0^1 (\sqrt{y} - y^2)^2 \, dy \]

\[ = \frac{\sqrt{3}}{4} \int_0^1 (y - 2y^{5/2} + y^4) \, dy \]

\[ = \sqrt{3} \left( \frac{1}{2} y^2 - \frac{4}{7} y^{7/2} + \frac{1}{5} y^5 \right) \bigg|_0^1 \]

\[ = \sqrt{3} \left( \frac{1}{2} - \frac{4}{7} + \frac{1}{5} \right) = \frac{9\sqrt{3}}{280} \]
Section 6.3: Volumes of Revolutions

Consider rotating an arbitrary region about the x-axis.

Marked region is rotated about the x-axis and we consider an arbitrary cross section perpendicular to the x-axis.
A typical cross section is a washer!

![Diagram of a washer with labels Rin, Rout, and cross-sectional area]

Area of washer:

\[ A_{\text{outer}} - A_{\text{inner}} = \pi Rout^2 - \pi Rin^2 \]

From the previous lecture, we know the volume of the solid is the integral of the cross-sectional area.

\[ V = \pi \int_a^b (Rout^2 - Rin^2) \, dh \]

\[ (V = \pi \int_a^b (f(x)^2 - g(x)^2) \, dx) \]

This is our formula for volume of revolution by Method of Washers

**Ex. 1**

The region bounded by \( y = x^2 + 4 \),
$y = 2$, $x = 1$, and $x = 3$ is rotated about the $x$-axis. Find the volume of the solid.

**Solution:**

$y = x^2 + 4$

Washers use slices that are perpendicular to the rotation axis.

$P = (x, x^2 + 4)$

$Q = (x, 2)$

$C = (x, 0)$

$R_{in} = |QC| = (2 - 0) = 2$

$R_{out} = |PC| = (x^2 + 4 - 0) = x^2 + 4$
So the volume is...

\[ V = \pi \int_{1}^{3} \left( (x^2 + 4)^2 - (2)^2 \right) dx \]

\[ = \pi \int_{1}^{3} (x^4 + 8x^2 + 12) dx = \frac{2126\pi}{15} \]

---

**Ex. 2**

Let \( R \) be the region bounded by \( y = x^2 + 2 \) and \( y = 4 - x^2 \). Find the volume of the solid obtained by rotating \( R \) about \( y = -3 \).

**Solution:**

[Diagram of the region \( R \) with points labeled: \((-1, 3), (1, 3), (2, 4), (2, 0), (-2, 0), (-2, 4)\), and the rotation about \( y = -3 \).]
\[ P = (x, 4-x^2) \]
\[ Q = (x, x^2+2) \]
\[ C = (x, -3) \]

\[ R_{\text{in}} = \int_{Q}^{C} = (x^2+2) - (-3) = x^2 + 5 \]
\[ R_{\text{out}} = \int_{P}^{C} = (4-x^2) - (-3) = 7 - x^2 \]

So our volume is...

\[ V = \pi \int_{-1}^{1} ((7-x^2) - (x^2+5))^2 \, dx = 32\pi \]

\[ R_{\text{out}}^2 - R_{\text{in}}^2 = (R_{\text{out}} - R_{\text{in}})(R_{\text{out}} + R_{\text{in}}) \]
\[ = (7-x^2-x^2-5)(12) \]
\[ = 12(2-2x^2) = 24(1-x^2) \]

---

**Ex. 3**

Let \( R \) be the region bounded by \( y = 2 + \sec(x) \) and \( y = 4 \) (for \( \frac{-\pi}{2} < x < \frac{\pi}{2} \)). Find the volume.
of the solid obtained by rotating $R$ about $y = 2$.

**Solution:**

Intersection points:

$4 + \sec(x) = 2 \implies \cos(x) = \frac{1}{2} \implies x = -\frac{\pi}{3}, \frac{\pi}{3}$

$P = (x, 4)$

$Q = (x, 2 + \sec(x))$

$C = (x, 2)$

$R_{in} = |\overline{QC}| = (2 + \sec(x)) - 2 = \sec(x)$

$R_{out} = |\overline{PC}| = 4 - 2 = 2$

So our volume is...
Let \( R \) be the region bounded by \( y = x \) and \( y = 2\sqrt{x} \). Find the volume of the solid obtained by rotating \( R \) about the line \( x = -2 \).

**Solution:**

\[
V = \pi \int_{-\pi/3}^{\pi/3} (R_{out}^2 - R_{in}^2) \, dx = \\
= \pi \int_{-\pi/3}^{\pi/3} (4 - \sec(x)^2) \, dx \\
= \pi \left( 4x - \tan(x) \right) \bigg|_{-\pi/3}^{\pi/3} = 2\pi \left( \frac{4\pi}{3} - \sqrt{3} \right)
\]
\[ 2\sqrt{x} = x \implies 4x = x^2 \implies 4x - x^2 = 0 \]
\[ \implies x(4-x) = 0 \implies x = 0 \text{ or } x = 4. \]

\[ P = (y, y) \]
\[ Q = \left(\frac{y^2}{4}, y\right) \]
\[ y = 2\sqrt{x} \implies x = \frac{y^2}{4} \]
\[ C = (-2, y) \]
\[ R_{in} = \left| \overline{QC} \right| = \left(\frac{y^2}{4}\right) - (-2) = \frac{y^2}{4} + 2 \]
\[ R_{out} = \left| \overline{PC} \right| = (y) - (-2) = y + 2 \]

So our volume is....

\[ V = \pi \int_{0}^{4} \left( R_{out}^2 - R_{in}^2 \right) dy \]
\[ = \pi \int_{0}^{4} \left( (y+2)^2 - \left(\frac{y^2}{4} + 2\right)^2 \right) dy \]
\[ = \pi \int_{0}^{4} \left( 4y - \frac{y^4}{16} \right) dy = \frac{96\pi}{5} \]

**Ex. 5**

A solid torus is obtained by rotating the disk.
\[(x-a)^2 + y^2 \leq b^2\]

about the y-axis \((a > b)\). Find the volume of the torus.

**Solution:**

\[P = (a + \sqrt{b^2 - y^2}, y)\]
\[Q = (a - \sqrt{b^2 - y^2}, y)\]
\[C = (0, y)\]

\[(x-a)^2 + y^2 = b^2\]
\[(x-a)^2 = b^2 - y^2\]
\[x - a = \pm \sqrt{b^2 - y^2}\]
\[ x = a \pm \sqrt{b^2 - y^2} \]

\[ R_{in} = |QC| = a - \sqrt{b^2 - y^2} \]
\[ R_{out} = |PC| = a + \sqrt{b^2 - y^2} \]

\[ R_{out}^2 - R_{in}^2 = (R_{out} - R_{in})(R_{out} + R_{in}) \]
\[ = (2\sqrt{b^2 - y^2})(2a) \]
\[ = 4a\sqrt{b^2 - y^2} \]

So our volume is...

\[ V = \pi \int_{-b}^{b} 4a\sqrt{b^2 - y^2} \, dy \]

\[ y = b \sin(\theta) \quad y = b \tanh(\theta) \]

\[ = 4a\pi \int_{-b}^{b} \sqrt{b^2 - y^2} \, dy \]

\[ \text{area under } u = \sqrt{b^2 - y^2} \]
\[ \text{from } y = -b \text{ to } y = b \]
\[ \text{(area of half-disc)} \]

\[ = 4a\pi \cdot \frac{\pi}{2} b^2 = 2\pi^2 ab^2 \]
Section 6.4: Method of Cylindrical Shells

This is not volume by slicing. Instead the volume is approximated by nested shells. So consider one thin shell:

inner radius: \( r_1 \)
outer radius: \( r_2 \)
average of inner and outer radius: \( \bar{r} = \frac{r_1 + r_2}{2} \)

When we say the shell is “thin” we mean the following:
1. thickness of shell \( r_2 - r_1 = dr \)

\( \Rightarrow \) \( dr \) is an arbitrarily small number
Even though the radius $r$ varies across the shell, we approximate $r$ as constant.

\[ \Rightarrow \text{so } r = \bar{r} \text{ for entire shell} \]

So now what is the volume of the shell?

Volume = (larger cylinder) - (smaller cylinder)

\[ dV = (\pi r_2^2 h) - (\pi r_1^2 h) \]

\[ dV = \pi h \cdot (r_2 + r_1)(r_2 - r_1) \]

\[ dV = 2\pi h \cdot \left( \frac{r_2 + r_1}{2} \right) \cdot (r_2 - r_1) = \bar{r} = dr \]

\[ dV = 2\pi h \bar{r} \, dr \]

The total volume of the solid is the "sum" of all the small volumes. But the "sum" of all $dV$'s is an integral.

**Total volume:** \[ V = \int dV = \int_0^R 2\pi h \bar{r} \, dr \]

(We will have $dr = dx$ or $dr = dy$, and $h$ and $\bar{r}$ depend on $x$ or $y$ accordingly.)
This is our formula for volume of revolution by Method of Shells:

**Summary:**
- \( h \) = height of shell
- \( \bar{r} \) = radius of shell
- \( dr \) = thickness of shell

**Ex. 1**

Let \( R \) be the region bounded by coordinate axes, \( x = 1 \), and

\[
f(x) = 1 - 2x + 3x^2 - 2x^3
\]

Find the volume of the solid obtained by rotating \( R \) about the y-axis.

**Solution:**

![Diagram of the region and function]

Why is method of washers very difficult?
Finding $f^{-1}(y)$ is essentially impossible. So we have to use shells.

Shells uses a strip parallel to the axis of rotation.

$P = (x, 0)$
$Q = (x, f(x)) = (x, 1 - 2x + 3x^2 - 2x^3)$
\[ h = \left| P Q I \right| = 1 - 2x + 3x^2 - 2x^3 \]

\( F = \text{“distance from axis to shell wall”} \]
\[ = x_{\text{wall}} - x_{\text{axis}} = x - 0 = x \]

(Generally, \( F = a \pm x \) or \( F = a \pm y \))

\[ dr = dx \]

(Vertical strip \( \Rightarrow dx \), Horizontal strip \( \Rightarrow dy \))

So the volume is

\[ V = 2\pi \int_{0}^{1} \left( 1 - 2x + 3x^2 - 2x^3 \right) x \, dx \]

\[ = 2\pi \left( \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{3}{4}x^4 - \frac{2}{5}x^5 \right) \Bigg|_{0}^{1} = \frac{11\pi}{30} \]

**Ex. 2**

Let \( R \) be region bounded by \( y = \sqrt{x} \) and \( y = x^2 \). Find volume of solid obtained by rotating \( R \) about y-axis.

**Solution:**
\[ P = (x, \sqrt{x}) \]
\[ Q = (x, x^2) \]
\[ h = |\overline{PQ}| = P_y - Q_y = \sqrt{x} - x^2 \]
\[ \bar{r} = x \]
\[ dr = dx \]
\[ V = 2\pi \int_0^1 h \bar{r} \, dr \]
\[ = 2\pi \int_0^1 (\sqrt{x} - x^2) \times d\bar{x} \]
\[ = 2\pi \int_0^1 (x^{3/2} - x^3) \, dx \]
\[ = 2\pi \left( \frac{2}{5} x^{5/2} - \frac{1}{4} x^4 \right) \bigg|_0^1 = \frac{3\pi}{10} \]
Washers

P = ( y², y ) \quad y = \sqrt{x} \implies x = y²

Q = ( \sqrt{y}, y ) \quad y = x² \implies x = \sqrt{y}

C = ( 0, y )

R_{out} = |QC| = Q_x - C_x = \sqrt{y}

R_{in} = |PC| = P_x - C_x = y²

V = \pi \int_0^1 (R_{out}² - R_{in}²) \, dy = \pi \int_0^1 (y - y⁴) \, dy

= \pi \left( \frac{1}{2}y² - \frac{1}{5}y⁵ \right) \bigg|_0^1 = \frac{3\pi}{10}

Should you use shells or washers?
Really you should determine whether \( dx \) or \( dy \) is easier. Then use shells or washers appropriately.

Washers: Slice \perp \) rotation axis
Shells: Strip \parallel \) rotation axis

So you decide orientation of slice or strip. Then you are forced to use one method.

---

**Ex. 3**

Let \( R \) be the region below:

Let \( S_x \) be the solid obtained by rotating \( R \) about \( x \)-axis.
(a) washers
(b) shells

Let $S_y$ be the solid obtained by rotating $R$ about $y$-axis.

(a) washers
(b) shells

**Solution:**

$x$-axis, Washers

---

$P = (x, 1)$

$Q = (x, \sin(x))$

$C = (x, 0)$

$R_{out} = |PC| = 1$
\[ R_{\text{in}} = \cos(C) = \sin(x) \]

\[ V = \pi \int_0^{\pi/2} (R_{\text{out}}^2 - R_{\text{in}}^2) \, dx \]

\[ V = \pi \int_0^{\pi/2} (1 - \sin(x)^2) \, dx \]

\[ V = \pi \int_0^{\pi/2} \cos(x)^2 \, dx \]

\[ u = \cos(x) \quad dv = \cos(x) \, dx \]

\[ du = -\sin(x) \, dx \quad v = \sin(x) \]

\[ V = \pi \left( \cos(x) \sin(x) \Big|_0^{\pi/2} + \int_0^{\pi/2} \sin(x)^2 \, dx \right) = 0 \]

\[ V = \pi \int_0^{\pi/2} \sin(x)^2 \, dx = \pi \int_0^{\pi/2} (1 - \cos(x)^2) \, dx \]

\[ V = \pi \int_0^{\pi/2} 1 \, dx - \pi \int_0^{\pi/2} \cos(x)^2 \, dx \]

\[ 2V = \pi \left( \frac{\pi}{2} \right) \Rightarrow V = \frac{\pi^2}{4} \]
\[ P = (0, y) \]
\[ Q = (\sin^{-1}(y), y) \]
\[ h = |PQ| = Q_x - P_x = \sin^{-1}(y) \]
\[ r = y \]
\[ dr = dy \]
\[ V = \int_{0}^{1} 2\pi h r \, dr = \int_{0}^{1} 2\pi y \sin^{-1}(y) \, dy \]

Substitution \( y = \sin(u) \), \( dy = \cos(u) \, du \) is one option but let's use IBP.
\[ u = \sin^{-1}(y) \quad dv = 2\pi y \, dy \]

\[ du = \frac{1}{\sqrt{1-y^2}} \, dy \quad v = \pi y^2 \]

\[ \pi y^2 \sin^{-1}(y) \bigg|_0^1 - \int_0^1 \frac{\pi y^2}{\sqrt{1-y^2}} \, dy \]

\[ (\pi \cdot 1 \cdot \frac{\pi}{2} - 0 = \frac{\pi^2}{2}) \]

\[ = \frac{\pi^2}{2} - \int_0^1 \frac{\pi y^2}{\sqrt{1-y^2}} \, dy \]

Method 1: Trig sub with \( y = \sin(\theta) \)

Method 2: Integration by parts

Look at integral separately.

\[ \int_0^1 \frac{\pi y^2}{\sqrt{1-y^2}} \, dy = \int_0^1 \pi y \cdot \frac{y}{\sqrt{1-y^2}} \, dy \]

\[ u = \pi y \quad dv = \frac{y}{\sqrt{1-y^2}} \, dy \]

\[ du = \pi dy \quad v = -\sqrt{1-y^2} \]
\[
\begin{align*}
&= -\pi y \sqrt{1 - y^2} \bigg|_0^1 + \int_0^1 \pi \sqrt{1 - y^2} \, dy \\
&= 0 \quad \text{area of a quarter-circle of radius 1.}
\end{align*}
\]

Going back to (*)

\[
V = \frac{\pi^2}{2} - \int_0^1 \frac{\pi y^2}{\sqrt{1 - y^2}} \, dy
\]

\[
= \frac{\pi^2}{2} - \left( 0 + \pi \cdot \frac{\pi}{4} \right) = \frac{\pi^2}{4}
\]

\text{y-axis, washers}

\[
C = P \quad Q \quad (\frac{\pi}{2}, 1)
\]

\[
P = (0, y) \quad Q = (\sin^{-1}(y), y) \quad C = (0, y)
\]

\[
\text{Rent} = |\overline{QC}| = Q_x - C_x = \sin^{-1}(y)
\]
\[ \text{Rin} = |\text{PCL}| = P_x - C_x = 0 \]

\[ V = \pi \int_0^1 (R_{out}^2 - R_{in}^2) \, dy = \pi \int_0^1 \sin^{-1}(y)^2 \, dy \]

There are a few ways to proceed:

**Method 1:**

Substitute \( u = \sin^{-1}(y) \), or \( y = \sin(u) \) (so \( dy = \cos(u) \, du \))

\[ V = \pi \int_0^1 \sin^{-1}(y) \, dy = \pi \int_0^{\pi/2} u^2 \cos(u) \, du \]

Now use IBP twice or tabular IBP

\[
\begin{align*}
& \quad u^2 \quad \Theta \quad \cos(u) \\
& 2u \quad \Theta \quad \sin(u) \\
& 2 \quad \Theta \quad -\cos(u) \\
& 0 \quad \Theta \quad -\sin(u)
\end{align*}
\]

\[ = \pi \left( u^2 \sin(u) + 2u \cos(u) - 2 \sin(u) \right) \bigg|_0^{\pi/2} \]

\[ = \pi \left[ \left( \frac{\pi^2}{4} \cdot 1 + 0 - 2 \right) - (0 + 0 - 0) \right] = \frac{\pi^3}{4} - 2\pi \]

**Method 2:**
Use IBP twice immediately

\[ u = (\sin^{-1}y)^2 \quad du = dy \]
\[ dv = \frac{2 \sin^{-1}y}{\sqrt{1-y^2}} \quad v = y \]

\[ V = \pi \int_0^1 \sin^{-1}(y)^2 dy = \]
\[ = \pi y \sin^{-1}(y)^2 \bigg|_0^1 - \int_0^1 \frac{2\pi y \sin^{-1}y}{\sqrt{1-y^2}} dy \]
\[ = (\pi \cdot 1 \cdot \frac{\pi^2}{4} - 0) = \frac{\pi^3}{4} \quad \text{Integration by parts again} \]

\[ u = 2\pi \sin^{-1}(y) \quad dv = \frac{y}{\sqrt{1-y^2}} dy \]
\[ du = \frac{2\pi}{\sqrt{1-y^2}} dy \quad v = -\sqrt{1-y^2} \]

\[ = \frac{\pi^3}{4} - \left( -2\pi \sqrt{1-y^2} \sin^{-1}(y) \bigg|_0^1 + \int_0^1 2\pi dy \right) \]
\[ = 0 - 0 = 0 \quad = 2\pi \]

\[ = \frac{\pi^3}{4} - 2\pi \quad \text{(Phew! Same as Method 1)} \]

\[ y-axis, shells \]
\[ P = (x, 1) \]
\[ Q = (x, \sin(x)) \]
\[ h = |PQ| = P_y - Q_y = 1 - \sin(x) \]
\[ \Gamma = x_{	ext{strip}} - x_{	ext{axis}} = x - 0 = x \]
\[ dr = dx \]
\[ V = 2\pi \int_0^{\pi/2} h \Gamma \, dr = 2\pi \int_0^{\pi/2} (1 - \sin(x)) x \, dx \]

Use IBP twice or tabular IBP:

\[
\begin{array}{c|c|c|c}
 x & + & 1 - \sin(x) \\
 1 & \rightarrow & x + \cos(x) \\
 0 & \rightarrow & \frac{1}{2} x^2 + \sin(x) \\
\end{array}
\]

\[
V = 2\pi \left[ x (x + \cos(x)) - 1 \left( \frac{1}{2} x^2 + \sin(x) \right) \right] \bigg|_0^{\pi/2} \\
= 2\pi \left[ \frac{1}{2} x^2 + x \cos(x) - \sin(x) \right] \bigg|_0^{\pi/2}
\]
Let $R$ be the region shown below. Find the volume of the solid obtained by rotating $R$ about $y = 8$.

**Solution:** Washers
\[ p = (x, \, 8 - 2x^2) \]
\[ Q = (x, \, 4 - x^2) \]
\[ C = (x, \, 8) \]
\[ R_{out} = |QC| = C_y - Q_y = 4 + x^2 \]
\[ R_{in} = |PC| = C_y - P_y = 2x^2 \]
\[ V = \pi \int_0^2 (R_{out}^2 - R_{in}^2) \, dx \]
\[ = \pi \int_0^2 \left( (4 + x^2)^2 - (2x^2)^2 \right) \, dx \]
\[ = \pi \int_0^2 \left( -3x^4 + 8x^2 + 16 \right) \, dx = \frac{512\pi}{15} \]

**Shells**

![Graph showing regions R1 and R2 with equations y = 4 - x^2 and y = 8 - 2x^2, integrated along the x-axis from 0 to 2.]
We need two integrals, one for rotating $R_1$ and one for rotating $R_2$.

**Integral for region $R_1$,**

\[ P = (\sqrt{4-y}, y) \quad y = 4 - x^2 \]

\[ Q = \left(\sqrt{\frac{8-y}{2}}, y\right) \quad y = 8 - 2x^2 \]

\[ h = |PQ| = Q_x - P_x = \sqrt{\frac{8-y}{2}} - \sqrt{4-y} \]

\[ r = 8 - y \]

\[ dr = dy \]

\[ V_{R_1} = 2\pi \int_0^4 \left(\sqrt{\frac{8-y}{2}} - \sqrt{4-y}\right)(8-y)\,dy \]
This integral is a lot of coefficients and arithmetic. Use substitution

\[ u = 4 - y, \quad -\, du = \, dy \]

\[ V_{R_1} = 2\pi \int_{4}^{0} \left( \frac{\sqrt{u+4}}{\sqrt{2}} - \sqrt{u} \right) (u+4) (-\, du) \]

\[ = 2\pi \int_{4}^{0} \left( \frac{(u+4)^{3/2}}{\sqrt{2}} - u^{3/2} - 4u^{1/2} \right) du = \ldots \ldots \]

(straightforward from there.)

Integral for region \( R_1 \),

\[ P = (0, y) \quad Q = \left( \sqrt{\frac{8-y}{2}}, \, y \right) \]

\( y = 8 - 2x^2 \)
\[ h = |PQ| = Q_x - P_x = \sqrt{\frac{8-y}{2}} \]

\[ r = 8-y \]

\[ dr = dy \]

\[ V_{R_2} = 2\pi \int_4^8 \left( \sqrt{\frac{8-y}{2}} \right) (8-y) dy \]

This integral is easier than the other.

\[ V_{R_2} = 2\pi \int_4^8 \frac{(8-y)^{3/2}}{\sqrt{2}} dy = \pi \sqrt{2} \int_4^8 (8-y)^{3/2} dy \]

\[ = -\frac{2}{5} \pi \sqrt{2} (8-y)^{5/2} \bigg|_4^8 = \ldots \]

With some arithmetic we get

\[ V_{R_1} + V_{R_2} = \frac{512\pi}{15} \]

(Washers is a lot easier!)
To find the length of a curve we approximate the curve using a polygonal curve (line segments). Consider a small line segment

\[ ds = \sqrt{(dx)^2 + (dy)^2} \]

When we say the segment is small, we mean that \( dx \) and \( dy \) are arbitrarily small. By Pythagorean Theorem,

\[
(ds)^2 = (dx)^2 + (dy)^2
\]

\[
ds = \sqrt{(dx)^2 + (dy)^2}
\]

\[
ds = \sqrt{(dx)^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)}
\]

\[
ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]
The total arc length is the “sum” of the small lengths:

\[ s = \int ds = \int_{a}^{b} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

This is our arc length formula.

**Note:** For a more careful derivation of this formula, see the supplementary notes.

We can also consider the length of a curve given as \( x = f(y) \).

\[ ds = \sqrt{(dx)^2 + (dy)^2} \]
\[ ds = \sqrt{(dy)^2 \left( \left( \frac{dx}{dy} \right)^2 + 1 \right)} \]
\[ ds = \sqrt{\left( \frac{dx}{dy} \right)^2 + 1} \, dy \]

\[ s = \int ds = \int_{c}^{d} \sqrt{\left( \frac{dx}{dy} \right)^2 + 1} \, dy \]

**Ex. 1**

Find arc length of graph of

\[ y = \frac{1}{12} x^3 + \frac{1}{x} \]
over the interval $1 \leq x \leq 3$.

Solution:

Use arc length formula.

$$1 + \left( \frac{dy}{dx} \right)^2 = 1 + \left( \frac{1}{4x^2} - \frac{1}{x^2} \right)^2$$

$$= 1 + \left( \frac{1}{16}x^4 + \frac{1}{x^4} - 2 \cdot \frac{1}{4}x \cdot \frac{1}{x^2} \right)$$

$$= 1 + \frac{1}{16}x^4 + \frac{1}{x^4} \odot \frac{1}{2} = \frac{1}{16}x^4 + \frac{1}{x^4} \odot \frac{1}{2}$$

$$= \left( \frac{1}{4}x^2 + \frac{1}{x^2} \right)^2$$ \text{ Perfect Square!}$$

$$S = \int_{1}^{3} \sqrt{ \left( \frac{1}{4}x^2 + \frac{1}{x^2} \right)^2 } \, dx$$

$$= \int_{1}^{3} \left( \frac{1}{4}x^2 + \frac{1}{x^2} \right) \, dx = \left( \frac{1}{12}x^3 - \frac{1}{x} \right) \bigg|_{1}^{3} = \frac{17}{6}$$
What if you are not clever? What if you do not recognize

\[ \frac{1}{16} x^4 + \frac{1}{x^4} + \frac{1}{2} \]
as a perfect square? There is (some) hope!

\[
\sqrt{1 + \left( \frac{dy}{dx} \right)^2} = \sqrt{\frac{1}{16} x^4 + \frac{1}{x^4} + \frac{1}{2}}
\]

\[
= \sqrt{\frac{x^8 + 16 + 8x^4}{16x^4}} = \sqrt{\frac{(x^4 + 4)^2}{(4x^2)^2}}
\]

\[
= \frac{x^4 + 4}{4x^2} = \frac{x^2}{4} + \frac{1}{x^2}
\]

So you can still proceed if you can recognize

\[ x^8 + 16 + 8x^4 \]
as a perfect square.

**Ex. 2**
Find the arc length of \( y = \cos(x) \) over the interval \( 0 \leq x \leq \frac{\pi}{2} \).

**Solution:**

\[
1 + \left( \frac{dy}{dx} \right)^2 = 1 + (-\sin x)^2 = 1 + \sin(x)^2
\]

\[
S = \int_0^{\frac{\pi}{2}} \sqrt{1 + \sin(x)^2} \, dx
\]

How do we do this?

This integral cannot be evaluated in terms of elementary functions! (So you need a computer!)

\[
S \approx 1.9101
\]

(c.f., elliptic functions)

---

**Ex. 3**

Find the length of the curve \( y = \frac{x^2}{2} \) over interval \( 0 \leq x \leq 1 \).

**Solution:**
$1 + \left( \frac{dy}{dx} \right)^2 = 1 + (x)^2 = 1 + x^2$

$s = \int_0^1 \sqrt{1 + x^2} \, dx$  \quad \text{How do we do this?}

\textbf{Trigonometric Sub: } x = \tan(\theta)

\textbf{Hyperbolic Sub: } x = \sinh(\theta)

So let's do \textbf{trigonometric substitution}!

\[ x = \tan(\theta) \quad x = 0 \implies \theta = 0 \]

\[ dx = \sec(\theta)^2 \, d\theta \quad x = 1 \implies \theta = \frac{\pi}{4} \]

\[ 1 + x^2 = \sec(\theta)^2 \]

\[ s = \int_0^{\pi/4} \sec(\theta) \sec(\theta)^2 \, d\theta = \int_0^{\pi/4} \sec(\theta)^3 \, d\theta \]

\[ u = \sec(\theta) \quad du = \sec(\theta)^2 \, d\theta \]

\[ du = \sec(\theta) \tan(\theta) \, d\theta \quad v = \tan(\theta) \]

\[ = \underbrace{\sec(\theta)\tan(\theta)}_0 \bigg|_0^{\pi/4} - \int_0^{\pi/4} \sec(\theta)\tan(\theta)^2 \, d\theta \]

\[ = (\sqrt{2} \cdot 1 - 0) = \sqrt{2} \]

\[ = \sec(\theta)^2 - 1 \]
\[ = \sqrt{2} - \int_{0}^{\pi/4} (\sec(\theta)^3 - \sec(\theta)) \, d\theta \]

\[ = \sqrt{2} - \left[ \int_{0}^{\pi/4} \sec(\theta)^3 \, d\theta + \int_{0}^{\pi/4} \sec(\theta) \, d\theta \right] = S \]

\[ = \sqrt{2} - S + \left[ \ln \left| \sec(\theta) + \tan(\theta) \right| \right]_{0}^{\pi/4} = (\ln(\sqrt{2}+1) - \ln(1+0)) = \ln(\sqrt{2}+1) \]

\[ = \sqrt{2} - S + \ln(\sqrt{2}+1) \]

So we have.....

\[ S = \sqrt{2} - S + \ln(\sqrt{2}+1) \]

\[ \Rightarrow S = \frac{\sqrt{2} + \ln(\sqrt{2}+1)}{2} \]

What if we had done hyperbolic?

\[ x = \sinh(\theta) \]

\[ dx = \cosh(\theta) \]

\[ 1 + x^2 = \cosh(\theta)^2 \]

\[ S = \int_{0}^{1} \sqrt{1+x^2} \, dx = \int_{\phi(1)}^{\phi(1)} \cosh(\theta)^2 \, d\theta \]
Easier? Maybe.

How much should you simplify?

Ex:

\[
\cosh \left( \sinh^{-1}(1) \right) = \sqrt{2}
\]

\[
\begin{align*}
\theta &= \sinh^{-1}(1) \\
\sinh(\theta) &= 1 \\
\cosh(\theta) &= \sqrt{2}
\end{align*}
\]

Ex:

\[
\sinh \left( \ln(3) \right) = \frac{3 - \frac{1}{3}}{2} = \frac{4}{3}
\]

\[
\sinh(x) = \frac{e^x - e^{-x}}{2}
\]

Ex. 4

Find arc length of \( y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 \) over the interval \( 0 \leq x \leq 1 \).

Solution:

\[
1 + \left( \frac{dy}{dx} \right)^2 = 1 + \left( \frac{3}{2} \cdot \frac{4\sqrt{2}}{3} x^{1/2} \right)^2 = 1 + 8x
\]

\[
S = \int_0^1 \sqrt{1 + 8x} \, dx = \left. \frac{2}{3} \cdot \frac{1}{8} (1 + 8x)^{3/2} \right|_0^1
\]
We consider rotating a graph about the x-axis. What is the area of the resulting surface? As with arc length, we approximate the curve by line segments. Consider rotating one small line segment. What shape do you get?

We get a conical band! (The surface of a frustum.) What is the area of the band?

\[ A = \pi (r_1 + r_2) l \]
Note: For a derivation of this formula, see the supplementary notes.

When we say the band is “thin” we mean the following:

1. height of band: $x_2 - x_1 = dx$
   $\Rightarrow$ $dx$ is an arbitrarily small number
   $\Rightarrow$ so $l = ds$ is arbitrarily small

2. Even though the radius $r$ varies across the band, we approximate $r$ as constant
   $\Rightarrow$ so $r = \bar{r}$ for entire band

So now what is the area of the band?

$\text{area} = \pi \cdot (\text{sum of radii}) \cdot (\text{slant height})$
The total area of the surface is the "sum" of all the small areas. But the "sum" of all dA's is an integral.

\[
\text{total area: } A = \int dA = \int_a^b 2\pi \bar{r} \, ds
\]

This is our formula for surface area of revolution.

Summary: \( \bar{r} = \) radius of band \\
\( ds = \) arc length along curve

Note: For a more careful derivation of this formula, see the supplementary notes.

Remember that \( ds \) is the formula for the arc length differential from earlier.

In this course we will consider surface area only for surfaces obtained by rotating \( y = f(x) \) about the x-axis!

In this case, we have

\[
dA = \pi \cdot (\bar{r}' + \bar{r}) \cdot ds \\
dA = 2\pi \bar{r} \, ds
\]
\[ F = f(x) \]
\[ ds = \sqrt{1 + f'(x)^2} \, dx \]

So the surface area formula gives

\[ A = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} \, dx \]

**Ex. 5**

Show that the surface area of a sphere is \( 4\pi R^2 \).

**Solution:**

We rotate the graph of

\[ y = \sqrt{R^2 - x^2} \quad (-R \leq x \leq R) \]

about \( x \)-axis. (This forms a sphere!)

So now let's use our surface area formula.

\[ 1 + \left( \frac{dy}{dx} \right)^2 = 1 + \left( \frac{-2x}{2\sqrt{R^2 - x^2}} \right)^2 \]

\[ = 1 + \frac{x^2}{R^2 - x^2} = \frac{R^2}{R^2 - x^2} \]
Find the area of the surface obtained by rotating $y = \sin(x)$ ($0 \leq x \leq \pi$) about the $x$-axis.

**Solution:**

\[
1 + \left( \frac{dy}{dx} \right)^2 = 1 + \left( \cos(x) \right)^2 = 1 + \cos(x)^2
\]

\[
A = \int_0^\pi 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

\[
= \int_0^\pi 2\pi \sin(x) \sqrt{1 + \cos(x)^2} \, dx
\]

Let $u = \cos(x)$, $du = -\sin(x) \, dx$

\[
= -\int_{\cos(0)}^{\cos(\pi)} 2\pi \sqrt{1+u^2} \, du
\]

\[
= \int_{\cos(\pi)}^{\cos(0)} 2\pi \sqrt{1+u^2} \, du
\]
Now put \( u = \tan(\theta) \) or \( u = \sinh(\theta) \) (See previous example for \( \int \sec(\theta)^3 d\theta \))

\[
= 2\pi \left( \sqrt{2} + \ln \left( \sqrt{2} + 1 \right) \right)
\]

**Ex. 7**

Find the area of the surface obtained by rotating the graph of

\[
y = \frac{1}{4}x^2 - \frac{1}{2} \ln(x) \quad (1 \leq x \leq e)
\]

about \( x \)-axis.

**Solution:**

\[
1 + \left( \frac{dy}{dx} \right)^2 = 1 + \left( \frac{1}{2}x - \frac{1}{2}x \right)^2 = 1 + \frac{x^2}{4} - 2 \cdot \frac{1}{2}x \cdot \frac{1}{2x} + \frac{1}{4x^2}
\]

\[
= 1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}
\]

\[
= \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} = \left( \frac{x}{2} + \frac{1}{2x} \right)^2
\]

\[
A = \int_1^e 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]
\[
A = \int_1^e 2\pi \left( \frac{1}{4} x^2 - \frac{1}{2} \ln(x) \right) \left( \frac{x}{2} + \frac{1}{2x} \right) \, dx
\]

\[
\frac{A}{2\pi} = \int_1^e \left( \frac{1}{8} x^3 + \frac{1}{8} x - \frac{1}{4} x \ln(x) - \frac{\ln(x)}{4x} \right) \, dx
\]

**A** (easy)

\[
\int_1^e \left( \frac{1}{8} x^3 + \frac{1}{8} x \right) \, dx = \left( \frac{x^4}{32} + \frac{x^2}{16} \right) \bigg|_1^e
\]

\[= \left( \frac{e^4}{32} + \frac{e^2}{16} \right) - \left( \frac{1}{32} + \frac{1}{16} \right) = \frac{e^4}{32} + \frac{e^2}{16} - \frac{3}{32}
\]

**B**

\[
\int_1^e \frac{1}{4} x \ln(x) \, dx =
\]

\[u = \ln(x) \quad du = \frac{1}{4} x \, dx
\]

\[dv = \frac{1}{x} \, dx \quad v = \frac{1}{8} x^2
\]

\[= \frac{1}{8} x^2 \ln(x) \bigg|_1^e - \int_1^e \frac{1}{8} x \, dx
\]

\[= \left( \frac{1}{8} x^2 \ln(x) - \frac{1}{16} x^2 \right) \bigg|_1^e = \frac{e^2}{16} + \frac{1}{16}
\]

**C**

**IBP**

**u-sub**
\( \int_1^e \frac{\ln(x)}{4x} \, dx = \int_0^1 \frac{u}{4} \, du = \frac{u^2}{8} \bigg|_0^1 = \frac{1}{8} \)

\( u = \ln(x) \), \( du = \frac{1}{x} \, dx \)

Putting this all together we get

\[
A = 2\pi \int_1^e \left( \frac{1}{8} x^3 + \frac{1}{8} x - \frac{1}{4} x \ln(x) - \frac{\ln(x)}{4x} \right) \, dx
\]

\[
= 2\pi \left( A - B - C \right) = \frac{\pi (e^4 - 9)}{16}
\]
Section 7.7: Improper Integrals

\[ \int_{a}^{b} f(x) \, dx \]

The Riemann integral requires that
- \( a \) and \( b \) are finite
- \( f(x) \) is bounded on \( [a, b] \).

(We generally assume \( f \) is continuous on \( [a, b] \), but we know the integral can handle jump discontinuities.)

If either of the above assumptions is dropped, we say that the integral is improper. We will consider four types of improper integrals:

1. \( a = -\infty \) OR \( b = \infty \)
2. both \( a = -\infty \) and \( b = \infty \)
3. \( f(x) \) has a vertical asymptote at \( x = a \) (but not \( x = b \)) OR \( f(x) \) has a vertical asymptote at \( x = b \) (but not \( x = a \))
4. \( f(x) \) has a vertical asymptote at \( x = c \), with \( a < c < b \).
In general, an improper integral is defined as a limit of proper Riemann integrals. The details are in the exercises below.

**Terminology:**

If \( \int_a^b f(x) \, dx \) exists (as a limit) we say the integral converges. Otherwise we say the integral diverges.

---

**Ex. 1**

Evaluate \( \int_0^\infty e^{-x} \, dx \).

**Solution:**

By definition,

\[
\int_0^\infty e^{-x} \, dx = \lim_{R \to \infty} \int_0^R e^{-x} \, dx = \lim_{R \to \infty} \left( -e^{-x} \right)_0^R
\]
\[
\lim_{R \to \infty} \left( 1 - e^{-R} \right) = 1 - 0 = 1
\]

**Bonus:**

\[
\int_{0}^{\infty} a^{-x} \, dx = \lim_{R \to \infty} \int_{0}^{R} a^{-x} \, dx = \lim_{R \to \infty} \left[ \frac{-a^{-x}}{\ln(a)} \right]_{0}^{R}
\]

\((a > 1)\)

\[
= \lim_{R \to \infty} \left( \frac{1 - a^{-R}}{\ln(a)} \right) = \frac{1}{\ln(a)}
\]

---

**Ex. 2**

Show that \(\int_{2}^{\infty} \frac{1}{x^3} \, dx\) converges.

**Solution:**

By definition,

\[
\int_{2}^{\infty} \frac{1}{x^3} \, dx = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x^3} \, dx = \lim_{b \to \infty} \left( \frac{-1}{2x^2} \right)_{2}^{b}
\]

\[
= \lim_{b \to \infty} \left( -\frac{1}{2b^2} + \frac{1}{8} \right) = \frac{1}{8} \quad \text{(converges!)}
\]

---

**Ex. 3**

Show that \(\int_{-\infty}^{-1} \frac{1}{x} \, dx\) diverges.
Solution:

By definition,

$$\int_{-\infty}^{-1} \frac{1}{x} \, dx = \lim_{a \to -\infty} \int_{a}^{-1} \frac{1}{x} \, dx = \lim_{a \to -\infty} \left( \ln(-x) \right|_{a}^{-1} \right)$$

$$= \lim_{a \to -\infty} \left( 0 - \ln(-a) \right) = \lim_{a \to -\infty} (-\ln(-a)) = -\infty$$

This integral diverges.

---

Ex. 4

Find \( \int_{-\infty}^{\infty} \frac{1}{4+x^2} \, dx \) or show it diverges.

Solution:

Since both ends of the interval are infinite, we must split the integral.

$$\int_{-\infty}^{\infty} \frac{1}{4+x^2} \, dx := \int_{-\infty}^{a} \frac{1}{4+x^2} \, dx + \int_{a}^{\infty} \frac{1}{4+x^2} \, dx$$

compute each separately

where \( a \) is a fixed number.

\( \lim_{R \to \infty} \int_{-R}^{R} \) is not okay!
Now each integral:

\[
\int_{-\infty}^{a} \frac{1}{4+x^2} \, dx = \lim_{b \to -\infty} \int_{b}^{a} \frac{1}{4+x^2} \, dx
\]

\[
= \lim_{b \to -\infty} \left( \frac{1}{2} \tan^{-1} \left( \frac{x}{2} \right) \right|_{b}^{a}
\]

\[
= \lim_{b \to -\infty} \left( \frac{1}{2} \tan^{-1} \left( \frac{a}{2} \right) - \frac{1}{2} \tan^{-1} \left( \frac{b}{2} \right) \right)
\]

\[
= \frac{1}{2} \tan^{-1} \left( \frac{a}{2} \right) + \frac{\pi}{4}
\]

This is not enough! Both integrals must converge for the original to converge. (If any one of the two split integrals diverges, the original diverges.)

\[
\int_{a}^{\infty} \frac{1}{4+x^2} \, dx = \lim_{b \to \infty} \left( \frac{1}{2} \tan^{-1} \left( \frac{x}{2} \right) \right|_{a}^{b}
\]

\[
= \lim_{b \to \infty} \left( \frac{1}{2} \tan^{-1} \left( \frac{b}{2} \right) - \frac{1}{2} \tan^{-1} \left( \frac{a}{2} \right) \right)
\[ = \frac{\pi}{4} - \frac{1}{2} \tan^{-1} \left( \frac{a}{2} \right) \text{ (finite)} \]

So the integral converges.

\[
\int_{-\infty}^{\infty} \frac{1}{4+x^2} \, dx = \frac{\pi}{2} \text{ (sum of two integrals)}
\]

Why not do the integral symmetrically?

\[
\int_{-\infty}^{\infty} x \, dx = \lim_{R \to \infty} \int_{-R}^{R} x \, dx = 0
\]

But.....

\[
\int_{0}^{R} x \, dx = \frac{R^2}{2} \to \infty \text{ as } R \to \infty
\]

\[
\int_{-R}^{0} x \, dx = -\frac{R^2}{2} \to -\infty \text{ as } R \to \infty
\]

So the integral diverges!
Show that \( \int_0^2 \frac{1}{\sqrt{4-x^2}} \, dx \) converges.

Solution:

This integral is improper because \( \frac{1}{\sqrt{4-x^2}} \) is unbounded near \( x = 2 \).

By definition,

\[
\int_0^2 \frac{1}{\sqrt{4-x^2}} \, dx = \lim_{a \to 2^-} \int_0^a \frac{1}{\sqrt{4-x^2}} \, dx
\]

\[
= \lim_{a \to 2^-} \left[ \sin^{-1} \left( \frac{x}{2} \right) \right]_0^a
\]

\[
= \lim_{a \to 2^-} \left( \sin^{-1} \left( \frac{a}{2} \right) - 0 \right) = \frac{\pi}{2}
\]

So the integral converges.

---

Ex. 6

Determine whether \( \int_{-2}^{1} \frac{1}{x^2} \, dx \) converges.
Solution:

Please do not do the following:

\[
\int_{-2}^{1} \frac{1}{x^2} \, dx = \left. -\frac{1}{x} \right|_{-2}^{1} = -\frac{1}{1} - \left(-\frac{1}{-2}\right) \\
= -1 - \frac{1}{2} = -\frac{3}{2} \quad \text{Yay!}
\]
(No! How can you get a negative #?)

Since \( \frac{1}{x^2} \) is unbounded near \( x=0 \), we have to split the integral

\[
\int_{-2}^{1} \frac{1}{x^2} \, dx = \int_{-2}^{0} \frac{1}{x^2} \, dx + \int_{0}^{1} \frac{1}{x^2} \, dx
\]

(A) \( \int_{-2}^{0} \frac{1}{x^2} \, dx = \lim_{a \to 0^-} \int_{-2}^{a} \frac{1}{x^2} \, dx \)

\[
= \lim_{a \to 0^-} \left( -\frac{1}{x} \right|_{-2}^{a} = \lim_{a \to 0^-} \left( -\frac{1}{a} + \frac{1}{2} \right) = \infty
\]
(We are done!)
③ \[ \int_{0}^{1} \frac{1}{x^2} \, dx = \lim_{a \to 0^+} \int_{a}^{1} \frac{1}{x^2} \, dx \]
\[= \lim_{a \to 0^+} \left( -1 + \frac{1}{a} \right) = \lim_{a \to 0^+} \left( -1 + \frac{1}{a} \right) = \infty \]
So \[ \int_{-2}^{1} \frac{1}{x^2} \, dx \text{ diverges.} \]

### Two Important Theorems

**Thm.** (p-test for integrals on \([a, \infty)\))

Suppose \( a > 0 \). Then

\[ \int_{a}^{\infty} \frac{1}{x^p} \, dx \text{ converges } \iff p > 1 \]

(diverges \( \iff p \leq 1 \))

**Proof:**

**Case I:** \( p = 1 \)

\[ \int_{a}^{\infty} \frac{1}{x} \, dx = \lim_{b \to \infty} \int_{a}^{b} \frac{1}{x} \, dx = \lim_{b \to \infty} \left[ \ln(x) \right]_{a}^{b} \]
\[= \lim_{b \to \infty} \left( \ln(b) - \ln(a) \right) = \infty \text{ (diverges)} \]
**Case II:** \( (p \neq 1) \)

\[
\int_0^\infty \frac{1}{x^p} \, dx = \lim_{b \to \infty} \int_a^b \frac{1}{x^p} \, dx
\]

\[
= \lim_{b \to \infty} \left( \frac{x^{1-p}}{1-p} \right)_{b=a}^{b} = \lim_{b \to \infty} \left( \frac{b^{1-p}}{1-p} \right) - \frac{a^{1-p}}{1-p}
\]

Is this finite?

\[
\lim_{b \to \infty} \left( b^{1-p} \right) = \begin{cases} 
\infty & \text{if } 1-p > 0 \\
0 & \text{if } 1-p < 0
\end{cases}
\]

So we see that if \( 1-p > 0 \) \((p < 1)\), we get divergence. If \( 1-p < 0 \) \((p > 1)\), then we get convergence. \( \square \)
**Thm:** (Comparison Test for Integrals) Suppose $f$ and $g$ are continuous with $f(x) \geq g(x) > 0$ for all $x \geq a$.

- If $\int_a^\infty f(x) \, dx$ converges, then $\int_a^\infty g(x) \, dx$ converges.
- If $\int_a^\infty g(x) \, dx$ diverges, then $\int_a^\infty f(x) \, dx$ diverges.

*“Smaller than convergent is convergent”*
*“Bigger than divergent is divergent”*

**Ex. 7.1**

Determine whether $\int_1^\infty \frac{1}{\sqrt{x^3+1}} \, dx$ converges.

**Solution:**

This cannot be done with FTC since we cannot get the antiderivative of $(x^3+1)^{-1}$. But we can use Comparison Test.
But what do we compare it to? If \( x \) is very large, 
\[
\frac{1}{\sqrt{x^3 + 1}} \sim \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}
\]
This suggests that we compare our integral to \( \int_1^\infty \frac{1}{x^{3/2}} \, dx \).

(Note that \( \int_1^\infty \frac{1}{x^{3/2}} \, dx \) converges since \( p = \frac{3}{2} > 1 \). So we would like it if our function were less than \( \frac{1}{x^{3/2}} \).)

\[
0 \leq x^3 \leq x^3 + 1
\]
\[
\frac{1}{x^3} \geq \frac{1}{x^3 + 1} \geq 0
\]
\[
\frac{1}{x^{3/2}} \geq \frac{1}{\sqrt{x^{3/2} + 1}} \geq 0
\]
So by the comparison test, 
\[
\int_1^\infty \frac{1}{\sqrt{x^3 + 1}} \, dx
\]
converges.
Determine whether $\int_0^\infty e^{-x^2}\,dx$ converges.

Solution:

Note: if $0 \leq x \leq 1$, then $0 \leq x^2 \leq x$

if $x > 1$, then $x^2 > x$

Let's use Comparison Test, with $e^{-x}$. We already know $\int_0^\infty e^{-x}\,dx$ converges from (Ex. 1). But there is a problem! The problem is that $e^{-x^2} \geq e^{-x}$ if $0 \leq x \leq 1$
which is not good for comparison test. But it doesn’t matter. All that matters is that $0 \leq e^{-x^2} \leq e^{-x}$ eventually!

Okay, let’s make this precise....

Note that $\int_0^1 e^{-x^2} \, dx$ converges since $[0, 1]$ is bounded and $e^{-x^2}$ is continuous. For $\int_1^\infty e^{-x^2} \, dx$, we observe that $\int_1^\infty e^{-x} \, dx$ converges and $e^{-x^2} \leq e^{-x}$ on $[1, \infty)$. So $\int_1^\infty e^{-x^2} \, dx$ converges by Comparison Test. Hence $\int_0^\infty e^{-x^2} \, dx$ converges.

Ex. 9

Determine whether $\int_0^\infty \frac{1}{5(x^3-1)^{1/4}} \, dx$ converges.

Solution:

Use Comparison Test.

(for $x$ very large)
\[
\frac{1}{(x^3-1)^{1/4}} \sim \frac{1}{(x^3)^{1/4}} \sim \frac{1}{x^{3/4}}
\]

Observe that \( \int_5^\infty \frac{1}{x^{3/4}} \, dx \) diverges by p-test \((p = \frac{3}{4} \leq 1)\). Now we have

\[
0 \leq x^3 - 1 \leq x^3 \quad \text{for } x \geq 5
\]

\[
0 \leq (x^3 - 1)^{1/4} \leq x^{3/4}
\]

\[
\frac{1}{(x^3-1)^{1/4}} \geq \frac{1}{x^{3/4}} \not< 0
\]

So \( \int_5^\infty \frac{1}{(x^3-1)^{1/4}} \, dx \) diverges by Comp. Test.
Section 10.1: Sequences

Def: A sequence \( \{a_n\}_{n=1}^{\infty} \) is a function from some subset of \( \mathbb{N} \) to \( \mathbb{R} \). We call \( a_n \) the \( n \)th term and \( n \) the index.

We think of sequences as lists. Sequences can be described in several ways.

Ex:

<table>
<thead>
<tr>
<th>General Term</th>
<th>Domain</th>
<th>Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( a_n = 1 - \frac{1}{n} )</td>
<td>( n \geq 1 )</td>
<td>( 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots )</td>
</tr>
<tr>
<td>2. ( a_n = 2^{-n} ) (geometric)</td>
<td>( n \geq 0 )</td>
<td>( 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots )</td>
</tr>
<tr>
<td>3. ( a_n = (-1)^n n )</td>
<td>( n \geq 0 )</td>
<td>( 0, -1, 2, -3, 4, \ldots )</td>
</tr>
</tbody>
</table>

alternating factor

4. \( a_n \) is the \( n \)th digit in the decimal expansion of \( \pi \) (explicit formula not given)
Consider the sequence
\[ a_n = \frac{1}{2} \left( a_{n-1} + \frac{2}{a_{n-1}} \right) \]
with \( a_0 = 1 \). What is the limit of this sequence \( (n \to \infty) \)?

Solution:
Look at the first few terms:
\[ a_0 = 1 \]
\[ a_1 = \frac{1}{2} \left( a_0 + \frac{2}{a_0} \right) = 1.5 \]
\[ a_2 = \frac{1}{2} \left( a_1 + \frac{2}{a_1} \right) = \frac{17}{12} \approx 1.4167 \]
\[
a_3 = \frac{1}{2} \left( a_2 + \frac{2}{a_2} \right) = \frac{577}{408} \approx 1.414216
\]

It looks like \( a_n \) has limit \( \sqrt{2} \).

Let's suppose \( \lim_{n \to \infty} a_n = L \) (exists).

\[
a_n = \frac{1}{2} \left( a_{n-1} + \frac{2}{a_{n-1}} \right) \quad (*)
\]

So take \( n \to \infty \) on both sides

\[
\lim_{n \to \infty} a_n = L, \quad \lim_{n \to \infty} a_{n-1} = L
\]

\[
\{a_n\}_{n=1}^{\infty} = \frac{1}{2} \left( a_0, a_1, a_2, \ldots \right) \text{ same limit!}
\]

\[
\{a_{n-1}\}_{n=1}^{\infty} = \frac{1}{2} \left( a_0, a_1, a_2, \ldots \right)
\]

So \( (*) \) becomes

\[
L = \frac{1}{2} \left( L + \frac{2}{L} \right) \Rightarrow L = \sqrt{2}
\]

But how do we know the limit exists in the first place?

(Bonus: If you apply Newton's Method to the function \( f(x) = x^2 - 2 \) with initial guess 1, you recover \( \{a_n\} \).)
**Def:** We say \( \{a_n\} \) **converges** to a limit \( L \) if \( \lim_{n \to \infty} a_n = L \). We write this also as “\( a_n \to L \)” (“\( A \)-en goes to \( L \)”.) This means for all \( \varepsilon > 0 \), there exists \( N \) such that if \( n \geq N \), then \( |a_n - L| < \varepsilon \).

---

**Basic Terminology**

- **Boundedness**

<table>
<thead>
<tr>
<th>Bounded from below</th>
<th>Bounded from above</th>
<th>Bounded</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m \leq a_n ) for all ( n )</td>
<td>( a_n \leq M ) for all ( n )</td>
<td>( m \leq a_n \leq M ) for all ( n )</td>
</tr>
</tbody>
</table>

- **Convergence**

<table>
<thead>
<tr>
<th>Converges</th>
<th>Diverges to Infinity</th>
<th>Diverges</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lim_{n \to \infty} a_n ) exists</td>
<td>( \lim_{n \to \infty} a_n = \pm \infty ) ( \lor ) ( \lim_{n \to \infty} a_n = -\infty )</td>
<td>( \lim_{n \to \infty} a_n ) does not exist and is not ( \pm \infty )</td>
</tr>
</tbody>
</table>
Monotonicity

<table>
<thead>
<tr>
<th>Monotonically Decreasing</th>
<th>Monotonically Increasing</th>
<th>Monotonic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{n+1} \leq a_n$ for all $n$</td>
<td>$a_{n+1} \geq a_n$ for all $n$</td>
<td>Either mono. decreasing OR mono. increasing</td>
</tr>
</tbody>
</table>

**Ex. 2**

For each sequence, identify which properties above are satisfied.

**Sequence**

$n = 1, 2, 3, 4, 5$

$a_n = 5 - \frac{1}{n}$

**Solution**

- Bounded from above
- Bounded from below
- Converges
- Monotonically increasing
- Monotonic

$a_n = (-1)^n$

- Bounded from above
- Bounded from below
- Diverges
Some Basic Theorems

The following theorems establish some basic relationships among the various properties described above.

Theorem 1:
- converges \[ \Rightarrow \text{ bounded} \]

Theorem 2:
- monotonically increasing AND bounded from above \[ \Rightarrow \text{ converges} \]
Theorem 3:

Monotonically decreasing AND bounded from below \implies \text{converges}

Theorem 4:

Let \( a_n = f(n) \). If \( x \in \mathbb{R} \) and \( \lim_{x \to \infty} f(x) = L \), then \( \lim_{n \to \infty} a_n = L \) also.

Careful! (Theorem 4):

If \( \lim_{x \to \infty} f(x) \) dne., theorem says nothing!

Ex: Let \( f(x) = \sin(\pi x) \). Then \( \lim_{x \to \infty} f(x) \) dne.
But \( \{a_n\}_{n=0}^{\infty} = \{0, 0, \ldots\} \), so \( a_n \to 0 \).

Theorem 5: (Squeeze Theorem)

Suppose \( a_n \to L \) and \( b_n \to L \). If \( a_n \leq c_n \leq b_n \) (for all \( n \))
then \( c_n \to L \) also.

Theorem 6:

If \( |a_n| \to 0 \), then \( a_n \to 0 \) also.
Proof:
Observe that \(-|a_n| \leq a_n \leq |a_n|\). Now use Theorem 5 (Squeeze Theorem).

**Careful! Theorem 6:**

We must have \(|a_n| \to 0\); any other limit does not allow the same conclusion. For instance, if \(|a_n| \to 1\), then \(|a_n^2|\) may or may not converge to 1.

**Ex:**

1. Let \(a_n = 1\). Then \(|a_n| \to 1\) and \(a_n \to 1\)
2. Let \(a_n = (-1)^n\). Then \(|a_n| = 1\) for all \(n\).
   
   So \(|a_n| \to 1\), but \(|a_n^2|\) diverges.

**Ex. 3**

Calculate the limit of each sequence.

(a) \(a_n = \left(1 + \frac{c}{n}\right)^n\) \((c = \text{const.})\)
(b) \(a_n = C^{1/n}\) \((C = \text{const.} > 0)\)
(c) \(a_n = n^{1/n}\)

**Solution:**

By Theorem 4, we can find the limits using
techniques of Calculus I (L'Hôpital's Rule)

(a) Let \( f(x) = \left(1 + \frac{c}{x}\right)^x \). If \( x \to \infty \), we have the expression "1\(^\infty\)". So put \( L = \lim_{x \to \infty} f(x) \).

\[
\ln(L) = \ln \left( \lim_{x \to \infty} \left(1 + \frac{c}{x}\right)^x \right) \\
= \lim_{x \to \infty} \left[ x \ln \left(1 + \frac{c}{x}\right) \right] \quad \text{"8\( \cdot \)0"} \\
= \lim_{x \to \infty} \left[ \frac{\ln \left(1 + \frac{c}{x}\right)}{1/x} \right] \quad \text{"0\( \cdot \)0"} \\
= \lim_{x \to \infty} \left[ \frac{1}{1 + c/x} \cdot \left(-\frac{c}{x^2}\right) \right] \\
= \lim_{x \to \infty} \left[ \frac{c}{1 + c/x} \right] = \frac{c}{1+0} = c
\]

So \( \ln(L) = c \), hence \( L = e^c \).

(For instance \( (1 - \frac{3}{n})^n \to e^{-3} \).)
(b) \( \lim_{n \to \infty} C^{1/n} = C^0 = 1 \)

(c) Put \( L = \lim_{x \to \infty} x^{1/x} \). Now we have

\[
\ln(L) = \lim_{x \to \infty} \ln(x^{1/x}) = \lim_{x \to \infty} \frac{\ln(x)}{x} \sim 0
\]

\[
H = \lim_{x \to \infty} \frac{1}{x} = 0
\]

So \( \ln(L) = 0 \), hence \( L = 1 \).

(For instance, \( \lim_{n \to \infty} n^{1/n^2} = \lim_{n \to \infty} (n^{1/n})^3 = 1 \).)

---

**Ex. 4**

Calculate the limit of \( a_n = \frac{(-1)^n}{\sqrt{n}} \).

**Solution**: We cannot go back to a real-variable function since \((-1)^x\) is ill-defined. Consider \( |a_n| \) instead.

\[
|a_n| = \left| \frac{(-1)^n}{\sqrt{n}} \right| = \frac{|(-1)^n|}{\sqrt{n}} = \frac{1}{\sqrt{n}} \to 0
\]
Since \( |a_n| \to 0 \), so does \( a_n \to 0 \).

**Geometric Sequence**

If \( a_n = r^n \) (where \( r \) is a constant), we call \( \{a_n\} \) a geometric sequence with **common ratio** \( r \).

The characterization of a geo. sequence:

\[
\frac{a_{n+1}}{a_n} = r = \text{const.} \quad (*)
\]

One more theorem...

**Theorem 7:**

If \( a_n = r^n \) with \( r = \text{const.} \), then

\[
\lim_{n \to \infty} a_n = \begin{cases} 
0 & \text{if } |r| < 1 \\
1 & \text{if } r = 1 \\
\infty & \text{if } r > 1 \\
dne. & \text{if } r \leq -1
\end{cases}
\]

**Proof:**

Let \( f(x) = r^x \). Then from Calculus I,
\[
\lim_{x \to \infty} r^x = \begin{cases} 
0 & \text{if } 0 \leq r < 1 \\
1 & \text{if } r = 1 \\
\infty & \text{if } r > 1
\end{cases}
\]

If \( r < 0 \), this does not apply since \( f(x) \) is not well-defined, so we know nothing about \( a_n = r^n \). If \(-1 < r < 0\), note that \( |r| < 1 \). So \( |r|^n \to 0 \). Hence by Theorem 6, \( r^n \to 0 \) also. If \( r < -1 \), the sequence \( r^n \) oscillates between larger and larger values of alternating sign. So \( \{r^n\} \) has no limit if \( r \leq -1 \). \[\square\]
Section 10.2: Summing an Infinite Series

Some motivation...

1. How do we calculate special numbers? $e$? $\pi$? $\sin(1)$?

   **Answer:**
   
   $e = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$

   $\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1}$

   $\sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!}$

2. How do we approximate functions with polynomials?

   $\sqrt{1+x}$? $e^x$? $\cos(x)$?

   (Calculus I tells you to use tangent line, but can we do better?)

   **Answer:**
   
   Linearization at $x=0$

   $\sqrt{1+x} = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 - \frac{5}{128} x^4 + \cdots$

   "tangent polynomial"??
\[ e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \ldots \]

\[ \sin(x) = x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \frac{1}{5040} x^7 + \ldots \]

(Note: Values of \( e^x \), \( \sin(x) \), \( \cos(x) \), and other exponential or logarithmic functions are today calculated by more efficient and advanced algorithms, but infinite sums can be used. Look up "CORDIC" on Wikipedia.)

... So what do these infinite sums even mean? How do we make sense of them?

**Basic Definitions and Terminology**

Consider the infinite sum (or series)

\[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots \]

We say \( a_n \) is the \( n \)th term of the series.

The \( N \)th partial sum of this series is

\[ S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \ldots + a_N \]
The value of the series is defined as the limit of the partial sums:

\[ \sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} S_N = \lim_{N \to \infty} (a_1 + a_2 + \ldots + a_N) \]

If this limit exists, we say the series converges. Otherwise we say the series diverges (or, if \( \lim_{N \to \infty} S_N = \infty \), diverges to infinity)

**Ex. 1**

Determine whether the series converges.

\[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots \]

**Solution:**

Look at partial sums:

\[ S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left( \frac{1}{n} - \frac{1}{n+1} \right) \]

Partial fraction decomposition

Write out some terms:
\[ S_N = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \ldots + \left( \frac{1}{N-1} - \frac{1}{N} \right) + \left( \frac{1}{N} - \frac{1}{N+1} \right) \]

Most of these terms cancel.

\[ S_N = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \ldots + \left( \frac{1}{N-1} - \frac{1}{N} \right) + \left( \frac{1}{N} - \frac{1}{N+1} \right) \]

All terms cancel except the first and last.

\[ S_N = 1 - \frac{1}{N+1} \]

(E.g., \( \sum_{n=1}^{4} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = 1 - \frac{1}{5} = \frac{4}{5} \))

Now by definition,

\[ \sum_{n=1}^{N} \frac{1}{n(n+1)} = \lim_{N \to \infty} S_N = \lim_{N \to \infty} \left( 1 - \frac{1}{N+1} \right) = 1 \]

So the series converges (has value 1).

Because of the cancellation in \( S_N \), we call this type of series a **telescoping series**.

---

**Ex. 2**

Find the value of \( \sum_{n=1}^{8} \frac{1}{n(n+3)} \).

**Solution:**
Use partial fraction decomposition.

\[ S_N = \sum_{n=1}^{N} \frac{1}{n(n+3)} = \sum_{n=1}^{N} \left( \frac{1/3}{n} - \frac{1/3}{n+3} \right) \]

\[ = \frac{1}{3} \sum_{n=1}^{N} \left( \frac{1}{n} - \frac{1}{n+3} \right) \]

Write out enough terms to see the pattern in cancellation.

\[ S_N = \frac{1}{3} \left[ \left( \frac{1}{1} - \frac{1}{4} \right) + \left( \frac{1}{2} - \frac{1}{5} \right) + \left( \frac{1}{3} - \frac{1}{6} \right) + \left( \frac{1}{4} - \frac{1}{7} \right) + \right. \]
\[ + \left( \frac{1}{5} - \frac{1}{8} \right) + \left( \frac{1}{6} - \frac{1}{9} \right) + \ldots + \left( \frac{1}{N-3} - \frac{1}{N} \right) + \left( \frac{1}{N-2} - \frac{1}{N+1} \right) + \]
\[ + \left( \frac{1}{N-1} - \frac{1}{N+2} \right) + \left( \frac{1}{N} - \frac{1}{N+3} \right) \]

How many terms "in front" do not cancel?

How many terms "in back" do not cancel?

\[ S_N = \frac{1}{3} \left[ \left( \frac{1}{1} - \frac{1}{4} \right) + \left( \frac{1}{2} - \frac{1}{5} \right) + \left( \frac{1}{3} - \frac{1}{6} \right) + \left( \frac{1}{4} - \frac{1}{7} \right) + \right. \]
\[ + \left( \frac{1}{5} - \frac{1}{8} \right) + \left( \frac{1}{6} - \frac{1}{9} \right) + \ldots \]
We are left with three uncancelled terms “in front”. We should get the same number of terms “in back”.

\[ S_N = \frac{1}{3} \left[ \ldots + \left( \frac{1}{N-3} - \frac{1}{N} \right) + \left( \frac{1}{N-2} - \frac{1}{N+1} \right) + \left( \frac{1}{N-1} - \frac{1}{N+2} \right) + \left( \frac{1}{N} - \frac{1}{N+3} \right) \right] \]

How do you know \( \frac{1}{N-3} \) cancels? Consider that

\[ S_N = a_1 + a_2 + \ldots + a_{N-6} + \ldots + a_{N-3} + \ldots + a_N \]

\[ = \left( \frac{1}{N-6} - \frac{1}{N-3} \right) = \left( \frac{1}{N-3} - \frac{1}{N} \right) \]

So one term partially cancels with another term three indices away!

Putting this together gives us:

\[ S_N = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{N+1} - \frac{1}{N+2} - \frac{1}{N+3} \]

So by definition we have

\[ \sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \lim_{N \to \infty} S_N = \]
\[
\lim_{N \to \infty} \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{N+1} - \frac{1}{N+2} - \frac{1}{N+3} \right) = \frac{11}{18}
\]

**Overall Picture: Sequences vs. Series**


<table>
<thead>
<tr>
<th>Sequence:</th>
<th>(a_1, a_2, a_3, \ldots, a_n, \ldots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limit:</td>
<td>(\lim_{n \to \infty} a_n = L)</td>
</tr>
<tr>
<td></td>
<td>(sequence converges if limit exists)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Partial sum sequence:</th>
<th>(S_1, S_2, S_3, \ldots, S_N, \ldots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((a_1, a_1+a_2, a_1+a_2+a_3, \ldots, \sum_{n=1}^{N} a_n, \ldots))</td>
<td></td>
</tr>
<tr>
<td>Limit:</td>
<td>(\lim_{N \to \infty} S_N = \sum_{n=1}^{\infty} a_n)</td>
</tr>
<tr>
<td></td>
<td>(series converges if limit exists)</td>
</tr>
</tbody>
</table>

So a series is defined in terms of a second sequence \(\{S_n\}\) which we derive from \(\{a_n\}\), the terms of the series.

**Q:** Is there a relationship between convergence of a sequence and convergence of the associated series?

**A:** Yes! (But be careful!)
Theorem 1: Nth-term Divergence Test
If \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{n=1}^{\infty} a_n \) diverges.

Proof:
We will prove the contrapositive: if \( \sum_{n=1}^{\infty} a_n \) converges, then \( a_n \to 0 \).

Suppose \( \sum_{n=1}^{\infty} a_n \) converges and let \( S_N \) be the Nth partial sum. Then
\[
a_N = (a_1 + \ldots + a_{N-1} + a_N) - (a_1 + \ldots + a_{N-1}) = S_N - S_{N-1}
\]

Now let \( N \to \infty \).

\[
\lim_{N \to \infty} a_N = \lim_{N \to \infty} (S_N - S_{N-1}) = \lim_{N \to \infty} S_N - \lim_{N \to \infty} S_{N-1}
\]

these limits exist!

they both equal \( \sum_{n=1}^{\infty} a_n \)

\[
= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} a_n = 0
\]

So \( a_n \to 0 \). \( \square \)

Ex. 3
Determine whether the series converges.
\[ \sum_{n=1}^{\infty} \frac{n^2 + 2}{4n(n+1)} \]

**Solution:**

Observe that

\[ \lim_{n \to \infty} \frac{n^2 + 2}{4n(n+1)} = \lim_{n \to \infty} \frac{1 + \frac{2}{n^2}}{4 + \frac{4}{n}} = \frac{1}{4} \]

Since the limit of the terms is not 0, the series diverges.

---

Be careful! The converse of Theorem 1 is false:

**FALSE:** “If \( a_n \to 0 \), then \( \sum_{n=1}^{\infty} a_n \) converges.”

If \( a_n \to 0 \), then \( \sum_{n=1}^{\infty} a_n \) may or may not converge. The next example shows that divergence can occur!

---

**Ex. 4**

Show that \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) diverges.

**Solution:**
(Note that \( \frac{1}{\sqrt{n}} \to 0 \), so the limit of the terms is 0. But this series diverges!) We can't find \( S_N \) explicitly but we can estimate it.

\[
S_N = \sum_{n=1}^{N} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{N}}
\]

Smallest term is \( \frac{1}{\sqrt{N}} \), so replacing all terms with \( \frac{1}{\sqrt{N}} \) makes sum smaller.

\[
S_N \gtrsim \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \ldots + \frac{1}{\sqrt{N}} = N \cdot \frac{1}{\sqrt{N}} = \sqrt{N}
\]

N total terms

So \( S_N \gtrsim \sqrt{N} \) for all \( N \). Now take \( N \to \infty \)

\[
\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \lim_{N \to \infty} S_N \gtrsim \lim_{N \to \infty} \sqrt{N} = \infty
\]

So the series diverges to \( \infty \).

---

**Theorem 2**: (Geometric Series)

Suppose \( c \) and \( r \) are constants with \( c \neq 0 \). Consider the geometric series
\[
\sum_{n=0}^{\infty} c \cdot r^n = c + cr + cr^2 + cr^3 + \ldots
\]
has the following sum:

\[
\sum_{n=0}^{\infty} c \cdot r^n = \begin{cases} 
\frac{c}{1-r} & \text{if } |r| < 1 \\
\text{diverges} & \text{if } |r| \geq 1
\end{cases}
\]

The formula for $|r| < 1$ should be memorized:

\[
\sum_{n=0}^{\infty} c \cdot r^n = \frac{(\text{first term})}{1 - (\text{common ratio})}
\]

**Proof**

**Case I: $r = 1$**

\[
S_N = \sum_{n=0}^{N} c \cdot 1^n = c + c + \ldots + c = (N+1)c
\]

$N+1$ total terms

So take $N \to \infty$. Since $c \neq 0$, $S_N \to \infty$ or $-\infty$. Either case, series diverges.

**Case II: $r \neq 1$**

Some very clever algebra!
\[ S_N = c + cr + cr^2 + \ldots + cr^N \]
\[ rS_N = \quad cr + cr^2 + \ldots + cr^N + cr^{N+1} \]
Subtract the equations. Cancellation!
\[ S_N - rS_N = c - cr^{N+1} \]
Solve algebraically for \( S_N \).
\[ S_N = \frac{c - cr^{N+1}}{1 - r} \]

Recall the result of Theorem 7 from section 10.1 lecture notes:
\[
\lim_{n \to \infty} r^n = \begin{cases} 
0 & \text{if } |r| < 1 \\
1 & \text{if } r = 1 \\
\infty & \text{if } r > 1 \\
dne. & \text{if } r \leq -1 
\end{cases}
\]

So now compute limit of \( S_N \).
\[
\lim_{N \to \infty} S_N = \lim_{N \to \infty} \frac{c - cr^{N+1}}{1 - r} = \begin{cases} 
\frac{c}{1 - r} & \text{if } |r| < 1 \\
\text{Case } r = 1 \text{ excluded} \\
diverges & \text{if } r > 1 \\
diverges & \text{if } r \leq -1 
\end{cases}
\]

So we recover the desired result. \( \square \)
Ex. 5

Calculate each of the following series or state the series diverges.

(a) \[ \sum_{n=0}^{8} 5^{-n} \]
(b) \[ \sum_{n=3}^{8} 7 \left(- \frac{3}{4}\right)^n \]
(c) \[ \sum_{n=1}^{8} e^{-2n} \]
(d) \[ \sum_{n=4}^{8} (-2)^n \]

Solution:

(a) \[ \sum_{n=0}^{8} 5^{-n} = \sum_{n=0}^{8} \left(\frac{1}{5}\right)^n = \frac{1}{1 - 1/5} = \frac{5}{4} \]

\[ r = \frac{1}{5} \]

\[ 5^{-n} = \frac{1}{5^n} = \left(\frac{1}{5}\right)^n \]

\[ \frac{a_{n+1}}{a_n} = \frac{5^{-(n+1)}}{5^{-n}} = \frac{5^{-n-1}}{5^{-n}} = 5^{-1} = \frac{1}{5} \]

(b) \[ \sum_{n=3}^{8} 7 \left(- \frac{3}{4}\right)^n = \frac{7 \left(- \frac{3}{4}\right)^3}{1 - \left(-\frac{3}{4}\right)} = -\frac{27}{16} \]

\[ r = -\frac{3}{4} \]

\[ \left[ \frac{\text{first term}}{1 - \text{common ratio}} \right] \]
\( (c) \sum_{n=1}^{\infty} e^{-2n} = \sum_{n=1}^{\infty} (e^{-2})^n = \frac{e^{-2}}{1-e^{-2}} \)

\[ \frac{a_{n+1}}{a_n} = \frac{e^{-2(n+1)}}{e^{-2n}} = \frac{e^{-2n-2}}{e^{-2n}} = e^{-2} \]

\( (d) \sum_{n=4}^{\infty} (-2)^n \quad \text{Since } |r| \geq 1, \text{ series diverges.} \]

\[ r = -2 \]

**Theorem 3**: (Linearity of Convergent Series)

Suppose \( \sum a_n \) and \( \sum b_n \) converge. Then
\[ \sum (a_n + b_n) = \sum a_n + \sum b_n \]
\[ \sum c \cdot b_n = c \cdot \sum b_n \]
(In particular, the series on the left converge.)

**Ex. 6**

Find the value of \( \sum_{n=2}^{\infty} \frac{3^n - 5^n}{7^n} \), or show the series diverges.

**Solution**: We would like to write
\[
\sum_{n=2}^{\infty} \frac{3^n - 5^n}{7^n} = \sum_{n=2}^{\infty} \frac{3^n}{7^n} - \sum_{n=2}^{\infty} \frac{5^n}{7^n}
\]

But this is valid only if each individual series converges.

\[
\sum_{n=2}^{\infty} \frac{3^n}{7^n} : r = \frac{3}{7} , \text{ so converges}
\]

\[
\sum_{n=2}^{\infty} \frac{5^n}{7^n} : r = \frac{5}{7} , \text{ so converges}
\]

(all we need is \(|r| < 1\).)

So our naive calculation was okay

\[
\sum_{n=2}^{\infty} \frac{3^n - 5^n}{7^n} = \sum_{n=2}^{\infty} \frac{3^n}{7^n} - \sum_{n=2}^{\infty} \frac{5^n}{7^n}
\]

\[
= \frac{(3/7)^2}{1 - 3/7} - \frac{(5/7)^2}{1 - 5/7} = -\frac{41}{28}
\]

---

Ex. 7

Write \(x = 0.219219219 \ldots\) as a ratio of two integers.

Solution:
By definition, this expansion means...

\[ x = 0.2190000000 \ldots + \]
\[ + 0.000219000 \ldots + \]
\[ + 0.000000219 \ldots + \]
\[ + \ldots \]

Write each term as a ratio of integers.

\[ x = \frac{219}{1000} + \frac{219}{1000^2} + \frac{219}{1000^3} + \ldots \]

So \( x \) is expressible as a convergent geometric series with:

- first term = \( \frac{219}{1000} \),
- common ratio = \( \frac{1}{1000} \)

\[ x = 0.219219219\ldots = \frac{219/1000}{1 - \frac{1}{1000}} = \frac{219}{999} \]

Internet trolls will claim that

\[ 0.999\ldots \neq 1 \leftarrow \text{FALSE!} \]

But now you know better!

\[ 0.999\ldots = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots = \frac{9/10}{1 - \frac{1}{10}} = 1 \]
**Final Note:** Be very careful when algebraically manipulating series. Many of the usual rules do not hold!

(This is why Theorem 3 requires the series to converge! Otherwise the theorem is false.)

As an example, consider the following:

\[ S = 1 + 2 + 4 + 8 + 16 + \ldots \]
\[ 2S = 2 + 4 + 8 + 16 + \ldots \]
\[ 2S + 1 = 1 + 2 + 4 + 8 + 16 + \ldots \]
\[ 2S + 1 = S \]

\[ \iff \text{ Algebra? } S = -1 ?? \]

How can \( S \) be negative if all its terms are positive?? The problem is that \( S \) is diverges to \( \infty \). So the last step:

“\( 2S + 1 = S \iff S = -1 \)”

is invalid because “\( 2S - S = \infty - \infty \)” is undefined.

(Why is \( S = \infty \)? \( S \) is geometric with \( r = 2 \).)
Section 10.3: Series with Positive Terms

Throughout this lecture, we will sum series with non-negative terms only.

\[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots \]

\[ a_n \geq 0 \quad \text{for all } n \]

**Theorem #1**

If \( a_n \geq 0 \) for all \( n \) with \( S_n = \sum_{n=1}^{N \geq} a_n \) and \( S = \lim_{n \to \infty} S_n \), then exactly one of the following is true:

1. \( S_n \) is bounded above, hence \( S \) converges.

   OR

2. \( S_n \) is not bounded above, hence \( S = \infty \).

**Proof:** Since \( a_n \geq 0 \), \( \{S_n\} \) is increasing. Now use Theorem #2 of Section 10.1.
Careful! Not true for general series.

$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

Partial sums are bounded, but series diverges (geometric with $r = -1$).

---

**Theorem #2 (Integral Test)**

Let $a_n = f(n)$ and suppose $f(x)$ is:

1. non-negative $\quad f(x) \geq 0$
2. decreasing $\quad x < y \implies f(x) > f(y)$
3. continuous $\quad f(x)$ is cont.

Then $\sum_{n=1}^{\infty} a_n$ converges $\iff \int_{1}^{\infty} f(x) \, dx$ converges.

**Proof:**

Consider the left and right Riemann sums for $\int_{1}^{N} f(x) \, dx$ by partitioning $[1, N]$ into $N$ equal subintervals.
Left Riemann sum:

\[ L_N = 1 \cdot f(1) + 1 \cdot f(2) + 1 \cdot f(3) + \ldots + 1 \cdot f(N-1) \]
\[ = a_1 + a_2 + a_3 + \ldots + a_{N-1} = S_{N-1} \]

Right Riemann sum:

\[ R_N = 1 \cdot f(2) + 1 \cdot f(3) + 1 \cdot f(4) + \ldots + 1 \cdot f(N) \]
\[ = a_2 + a_3 + a_4 + \ldots + a_N = S_N - a_1 \]

Since \( f(x) \) is decreasing we have:

\[ R_N \leq \int_1^N f(x) \, dx \leq L_N \]
Hence we have, for all $N$:

$$S_N - a_1 \leq \int_1^N f(x)\,dx \leq S_{N-1}$$

- Suppose $\int_1^\infty f(x)\,dx$ converges. Then $S_N - a_1 \leq \int_1^N f(x)\,dx \leq \int_1^\infty f(x)\,dx$. So $S_N \leq a_1 + \int_1^\infty f(x)\,dx$ for all $N$. Hence $\{S_N\}$ is bounded above. By Theorem #1, $\lim_{N \to \infty} S_N = \sum_{n=1}^\infty a_n$ converges since $a_n \geq 0$ for all $n$.

- Suppose $\int_1^\infty f(x)\,dx$ diverges. Then the sequence $b_N = \int_1^N f(x)\,dx$ is increasing (since $f(x) \geq 0$) and grows without bound. Since $\int_1^N f(x)\,dx \leq S_{N-1}$ for all $N$, $\{S_{N-1}\}$ is not bounded above. Hence by Theorem #1, $\lim_{N \to \infty} S_N = \infty$, so $\sum_{n=1}^\infty a_n$ diverges.
Note: If $f(x)$ satisfies the hypotheses of the Integral Test, then we always have

$$S_N - a_1 \leq \int_1^N f(x) \, dx \leq S_{N-1}$$

... or, equivalently...

$$f(N) + \int_1^N f(x) \, dx \leq S_N \leq f(1) + \int_1^N f(x) \, dx$$

This is true even if the series and integral diverge. This inequality can be used to estimate convergent sums or to estimate how fast a divergent series grows.

**Ex:** Let $S_N = \sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$.

Then we have from the above inequality:

$$\frac{1}{N} + \int_1^N \frac{1}{x} \, dx \leq S_N \leq 1 + \int_1^N \frac{1}{x} \, dx$$

$$\frac{1}{N} + \ln(N) \leq S_N \leq 1 + \ln(N)$$

$S_N$ grows at the same rate as $\ln(N)$.
Note: You are not responsible for knowing this approximation that comes out of the proof of Integral Test. But it is interesting nonetheless!

Ex. 1

Determine convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

(Harmonic series.)

Solution:

Let $f(x) = \frac{1}{x}$. Use Integral Test. Observe the following:

1. $f(x) > 0$ (for $x > 0$)

2. $f(x)$ is decreasing

$$f'(x) = -\frac{1}{x^2} < 0 \quad (\text{for all } x)$$

So $f$ is decreasing on $(0, \infty)$
③ f(x) is cont. since it is diff.
Now we have
\[ \int_1^\infty \frac{1}{x} \, dx = \lim_{R \to \infty} \ln (x) \bigg|_1^R = \lim_{R \to \infty} \ln (R) = \infty \]

Since integral diverges, series diverges.

Theorem #3 (p-test)
The series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges \( \iff \) \( p > 1 \).

Proof:
\( p > 0 \): Use Integral Test and p-test for Integrals from Section 7.7 notes.
\( p \leq 0 \): Use Nth-term divergence test. \( \square \)

Ex. 2
Determine whether series converges.
\[ \sum_{n=1}^{\infty} n e^{-n^2} \]

Solution:
Use Integral Test. Let \( f(x) = xe^{-x^2} \).
Observe the following:

1. \( f(x) > 0 \) (for \( x > 0 \))

2. \( f \) is decreasing

\[
f''(x) = x \cdot e^{-x^2} \cdot (-2x) + e^{-x^2} \cdot (1) \]

\[
f'(x) = e^{-x^2} (1 - 2x^2)
\]

Note that \( f'(x) < 0 \) for \( 1 - 2x^2 < 0 \), or \( x > \frac{1}{\sqrt{2}} \). So \( f \) is decreasing on \((1, \infty)\).

3. \( f \) is cont. since it is diff.

Now we have

\[
\int_{1}^{\infty} x e^{-x^2} \, dx = \lim_{R \to \infty} \left( -\frac{1}{2} e^{-x^2} \right)_{1}^{R}
\]

\[
= \frac{1}{2e} - \lim_{R \to \infty} \frac{1}{2eR^2} = \frac{1}{2e}
\]

Integral converges, so the series converges.
Reminder: The value of $\sum_{n=1}^{\infty} n e^{-n^2}$ is not $\frac{1}{2e}$. All we know from the earlier approximation is that
\[\int_1^{\infty} xe^{-x^2} \, dx \leq \sum_{n=1}^{\infty} n e^{-n^2} \leq f(1) + \int_1^{\infty} xe^{-x^2} \, dx\]
\[\Rightarrow \frac{1}{2e} \leq \sum_{n=1}^{\infty} n e^{-n^2} \leq \frac{1}{e} + \frac{1}{2e}\]

**Theorem #4:** (Direct Comparison Test / DCT)
Suppose $0 \leq a_n \leq b_n$ for all $n$.

(a) If $\sum b_n$ converges, then $\sum a_n$ converges.

(b) If $\sum a_n$ diverges, then $\sum b_n$ diverges.

**Ex. 3**
Determine whether series converges:
\[ \sum_{n=1}^{\infty} \frac{1}{n \cdot 4^n} \]

**Solution:**

Observe the following: (for all \( n \geq 1 \))

\[ \frac{1}{n \cdot 4^n} \leq \frac{1}{n} \]

\[ \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges by p-test (} p = 1 \geq 1 \text{)} \]

\[ \frac{1}{n \cdot 4^n} \leq \frac{1}{4^n} \]

\[ \sum_{n=1}^{\infty} \frac{1}{4^n} \text{ converges by geometric series test with } r = \frac{1}{4} \text{ and } |r| < 1. \]

(The first inequality is useless. "Smaller than divergent" means nothing.) So...

Observe that \( \frac{1}{n \cdot 4^n} \leq \frac{1}{4^n} \) and \( \sum_{n=1}^{\infty} \frac{1}{4^n} \text{ converges by Geometric Series Test, with } r = \frac{1}{4}. \) So by DCT, \( \sum_{n=1}^{\infty} \frac{1}{n \cdot 4^n} \text{ converges} \)

Determine whether series converges.

\[ \sum_{n=4}^{\infty} \frac{1}{(n^2 - 3)^{1/3}} \]

**Solution:**

Observe that

\[ 0 \leq n^2 - 3 \leq n^2 \quad \text{for } n \geq 4 \]

\[ 0 \leq \frac{1}{n^{2/3}} \leq \frac{1}{(n^2 - 3)^{1/3}} \]

The series \( \sum_{n=4}^{\infty} \frac{1}{n^{2/3}} \) diverges by \( p \)-test \( (p = \frac{2}{3} \leq 1) \). So by DCT, \( \sum_{n=4}^{\infty} \frac{1}{(n^2 - 3)^{1/3}} \) diverges.

---

**Theorem #5 (Limit Comparison Test / LCT):**

Suppose \( a_n > 0 \) and \( b_n > 0 \) for all \( n \). Let
\[ L = \lim_{n \to \infty} \frac{a_n}{b_n} \]

1. If \(0 < L < \infty\), then \(\sum a_n\) and \(\sum b_n\) both converge or both diverge.

2. If \(L = 0\) and \(\sum b_n\) converges, then \(\sum a_n\) converges also.

3. If \(L = \infty\) and \(\sum b_n\) diverges, then \(\sum a_n\) diverges also.

**Ex. 5**

Determine convergence of

\[
\sum_{n = 4}^{\infty} \frac{1}{(n^2 + 3)^{1/2}}
\]

**Solution:**

DCT is difficult to use since

\[
0 \leq \frac{1}{(n^2 + 3)^{1/2}} \leq \frac{1}{n^{2/3}}
\]

Use LCT instead. Observe that:
L = \lim_{n \to \infty} \frac{1}{(n^2 + 3)^{\frac{1}{3}}} \\
= \lim_{n \to \infty} \frac{n^{\frac{2}{3}}}{(n^2 + 3)^{\frac{1}{3}}} = \lim_{n \to \infty} \left( \frac{n^2}{n^2 + 3} \right)^{\frac{1}{3}} \\
= \lim_{n \to \infty} \left( \frac{1}{1 + \frac{3}{n^2}} \right)^{\frac{1}{3}} = \left( \frac{1}{1 + 0} \right)^{\frac{1}{3}} = 1

The series \( \sum_{n=4}^{\infty} \frac{1}{n^{2/3}} \) diverges by the \( p \)-test (\( p = 2/3 \leq 1 \)). So, since \( 0 < L < \infty \), by LCT, \( \sum_{n=4}^{\infty} \frac{1}{(n^2 + 3)^{\frac{1}{3}}} \) diverges.

**Ex. 6**

Determine convergence of
\[
\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n^9 + n + 1)^{\frac{1}{4}}}
\]
Solution:

If $a_n$ is an algebraic function of $n$, use LCT combined with $p$-test.

If $n$ is very large,

$$\frac{\sqrt{n}}{(n^q + n + 1)^{1/4}} \sim \frac{n^{1/2}}{(n^q)^{1/4}} \sim \frac{1}{n^{7/4}}$$

Now use LCT.

$$L = \lim_{n \to \infty} \frac{\sqrt{n}}{(n^q + n + 1)^{1/4}} = \frac{1}{n^{7/4}}$$

$$= \lim_{n \to \infty} \frac{n^{9/4}}{(n^q + n+1)^{1/4}} = \lim_{n \to \infty} \left(\frac{n^q}{n^q + n+1}\right)^{1/4}$$

$$= \lim_{n \to \infty} \left(\frac{1}{1 + \frac{1}{n^8} + \frac{1}{n^q}}\right)^{1/4} = 1$$
The series \( \sum_{i=1}^{\infty} \frac{1}{n^{7/4}} \) converges by p-test \((p = 7/4 > 1)\). Since \(0 < L < \infty\) by \(LCT\), \(\sum_{i=1}^{\infty} \frac{\sqrt{n}}{(n^q + n+1)^{1/4}}\) converges.

Ex. 7

Determine convergence of each series:

(a) \( \sum_{i=2}^{\infty} \frac{\ln(n)}{n} \)

(b) \( \sum_{i=2}^{\infty} \frac{1}{n^2 \ln(n)} \)

(c) \( \sum_{i=2}^{\infty} \frac{\ln(n)}{n^2} \)

(d) \( \sum_{i=2}^{\infty} \frac{1}{n \ln(n)} \)

Solution:

(a) Use DCT. 

\( \ln(n) > 1 \) (for \( n > \text{e} \))

\( \frac{\ln(n)}{n} > \frac{1}{n} \)

\( \sum \frac{1}{n} \) diverges by p-test \((p=1 \leq 1)\)
\[ \sum \frac{\ln(n)}{n} \text{ diverges by DCT} \]

(b) Use DCT.

\[
\ln(n) > 1 \quad (\text{for } n \geq 3)
\]

\[
\frac{1}{\ln(n)} < 1
\]

\[
\frac{1}{n^2 \ln(n)} < \frac{1}{n^2}
\]

\[ \sum \frac{1}{n^2} \text{ converges by p-test (} p = 2 > 1) \]

\[ \sum \frac{1}{n^2 \ln(n)} \text{ converges by DCT.} \]

(c) Note that

\[
\frac{\ln(n)}{n^2} > \frac{1}{n^2}
\]

This inequality is useless since \[ \sum \frac{1}{n^2} \]

converges. What about LCT?

\[
L = \lim_{n \to \infty} \frac{\ln(n)/n^2}{1/n^2} = \lim_{n \to \infty} \ln(n) = \infty
\]
This is also useless; $\sum \frac{1}{n^2}$ converges but $L=\infty$ is good only for testing divergence!

**General hierarchy:**
For any fixed $a > 0$, we have

$$\ln(n) < n^a$$

for large enough $n$.

So we can use DCT or LCT, but not with $\frac{1}{n^2}$. For large enough $n$,

$$\frac{\ln(n)}{n^2} < \frac{n^a}{n^2} = \frac{1}{n^{2-a}}$$

So just choose $a$ so that $a > 0$ and $2-a > 1$ (to get convergence). That is, choose a so $0 < a < 1$. What does this mean? Use DCT or LCT, but compare to $\sum \frac{1}{np}$ where $1 < p < 2$. 
\[
L = \lim_{n \to \infty} \frac{\ln(n)/n^2}{1/n^{1.5}} = \lim_{n \to \infty} \frac{\ln(n)}{n^{0.5}}
\]
\[
H = \lim_{n \to \infty} \frac{1/n}{0.5n^{-0.5}} = \lim_{n \to \infty} \frac{2}{n^{1.5}} = 0
\]

Observe that \(\sum \frac{1}{n^{1.5}}\) converges by p-test (\(p = 1.5 > 1\)). Since \(L = 0\), \(\sum \frac{\ln(n)}{n^2}\) converges by LCT.

(d) We have a similar problem here.
\[
\frac{1}{n \ln(n)} \leq \frac{1}{n} \quad \text{(for } n \geq 3\text{)}
\]
The inequality is in the wrong direction. Use Integral Test! Let \(f(x) = \frac{1}{x \ln(x)}\).

- \(f(x) \geq 0\) (obvious)
- \(x\) is increasing, \((n(x))\) is increasing
  \(x \ln(x)\) is increasing
- \(\frac{1}{x \ln(x)}\) is decreasing
\( f(x) \) is continuous (for \( n \geq 3 \))

Now we compute the integral.

\[
\int_{2}^{\infty} \frac{1}{x \ln(x)} \, dx = \int_{\ln(2)}^{\infty} \frac{1}{u} \, du = \ln(u) \bigg|_{\ln(2)}^{\infty}
\]

\( u = \ln(x), \, du = \frac{1}{x} \, dx \)

\[
= \lim_{R \to \infty} \ln(u) \bigg|_{\ln(2)}^{R} = \lim_{R \to \infty} \left( \ln(R) - \ln(\ln(2)) \right) = \infty
\]

Since the integral diverges, the series diverges by Integral Test.
Definition:
If \( \sum|a_n| \) converges, we say \( \sum a_n \) converges absolutely.

Theorem #1 (Absolute Convergence Test)
If \( \sum a_n \) converges absolutely, then \( \sum a_n \) converges.

Proof:
We always have \(-|a_n| \leq a_n \leq |a_n|\), and so
\[
0 \leq a_n + |a_n| \leq 2|a_n|
\]
If \( \sum |a_n| \) converges, then \( \sum 2|a_n| \) converges, and so \( \sum (a_n + |a_n|) \) converges by DCT.
Now we have
\[
\sum (a_n + |a_n|) - \sum |a_n| = \sum a_n
\]
hence, this converges also.
Definition:
If \( \sum a_n \) converges but not \( \text{absolutely} \) (i.e., \( \sum |a_n| \) diverges), we say \( \sum a_n \) conditionally converges.

Ex. 1

Determine convergence of
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = (-\frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots).
\]

Solution:
Note that \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges by \( p \)-test \( (p=2>1) \). Hence \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \) converges absolutely, so it converges.

Ex. 2

Determine convergence of
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.
\]
Solution: Note that \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges by p-test. This tells us nothing about \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \). So we need a new test!

Theorem #2 (Alternating Series Test / AST)
Suppose \( \{a_n\} \) is:
(a) \( a_n > 0 \)
(b) \( \{a_n\} \) is decreasing \( (a_n > a_{n+1}) \)
(c) \( a_n \to 0 \)
Then \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{a_n} \) converges.

Proof:
The intuition is that the partial sums zigzag around the value of \( \sum_{n=1}^{\infty} (-1)^{n-1} a_n \). So put
\[
S_m = \sum_{n=1}^{m} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{m-1} a_m
\]

Now consider a plot of \( \{S_m\} \).
\[ S_1 = a_1 \geq 0 \]
\[ S_2 = S_1 - a_2 \leq S_1 \quad \text{(since } a_2 \geq 0) \]
\[ S_3 = S_2 + a_3 \]
\[ \text{is this smaller or larger than } S_1? \]
\[ = a_1 - a_2 + a_3 = a_1 - (a_2 - a_3) \leq a_1 = S_1 \]
\[ \geq 0 \text{ since } \{a_n\} \text{ is decreasing} \]
\[ S_4 = S_2 + a_3 - a_4 \geq S_2 \]
\[ \geq 0 \text{ since } \{a_n\} \text{ is decreasing} \]

Let's make this intuition more precise.

(i) (Even partial sums)
\[ S_{2N+2} = S_{2N} + a_{2N+1} - a_{2N+2} \geq S_{2N} \]
\[ \geq 0 \text{ since } \{a_n\} \text{ is decreasing} \]
\[ S_2 \leq S_4 \leq S_6 \leq \ldots. \]

2) (Odd partial sums)

\[
S_{2N+1} = S_{2N-1} - a_{2N} + a_{2N+1} \\
= S_{2N-1} - (a_{2N} - a_{2N+1}) \leq S_{2N-1} \]

\[ \geq 0 \text{ since } a_n \text{ is decreasing} \]

\[ \ldots \leq S_5 \leq S_3 \leq S_1 \]

3) So we have

\[ S_2 \leq S_4 \leq S_6 \leq \ldots \leq S_5 \leq S_3 \leq S_1 \]

Odd partial sums: decreasing, bounded below

Even partial sums: increasing, bounded above

So by Theorem #3 and Theorem #4 of the Section 10.1 notes,

\[ \lim_{N \to \infty} S_{2N+1} = \text{Odd} \quad \text{and} \quad \lim_{N \to \infty} S_{2N} = \text{Even} \]

4) Now we must show that these two
limits are equal. (Why? There could be a gap in the zigzagging!)

\[
\text{S odd} - \text{S even} = \lim_{N \to \infty} (S_{2N+1} - S_{2N}) \quad \text{by assumption}
\]

\[
= \lim_{N \to \infty} (a_{2N+1}) = 0
\]

So \( \text{S odd} = \text{S even} \), and \( \lim_{N \to \infty} S_N \) exists. \( \square \)

The proof of AST gives us a way to approximate the value of such a series.

**Theorem #3 (AST Approximation Theorem)**

Suppose \( \{a_n\} \) satisfies the hypotheses of the AST. Then

\[
|S_N - S| \leq a_{N+1}, \quad \text{for all } N
\]

That is, if we approximate \( S \) by \( S_N \), the error is at most the absolute value of the first omitted term.

**Ex. 2** (Revisited)
Determine convergence of
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \]

Solution:
Let \( a_n = \frac{1}{n} \). Observe:

(a) \( a_n > 0 \) for \( n \geq 1 \) \( \{ \) obvious \( \}
(b) \( \{ a_n \} \) is decreasing
(c) \( a_n \to 0 \)

So by AST, \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \) converges.

Ex. 3
For the previous example....
(a) What is the largest error possible if we estimate \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \) using the first four terms? Over- or under-estimate?
(b) How many terms do you need to guarantee of at most \( 10^{-16} \)?
Solution:

(a) We have

\[ S = \sum_{n=1}^{8} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \]

= \( S_4 \), our approximation of \( S \)

Look at first omitted term to solve the problem:

error between \( S_4 \) and \( S \)

is at most \( \frac{1}{5} \), or 0.2

this term would increase the overall sum, so \( S_4 \) is an underestimate of \( S \).

(b) We seek the smallest value of \( N \) such that

\[ a_{N+1} \leq 10^{-16} \]

\[ \frac{1}{N+1} \leq 10^{-16} \]

\[ N \geq 10^{16} - 1 \]

So we need at least \( 9,999,999,999,999,999 \)
terms to guarantee error at most $10^{-16}$.

**Ex. 4**

Determine convergence of

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)} = \frac{1}{\ln(2)} - \frac{1}{\ln(3)} + \frac{1}{\ln(4)} - \ldots$$

How many terms do you need to guarantee error at most $10^{-16}$?

**Solution:**

Let $a_n = \frac{1}{\ln(n)}$. Observe:

- $a_n > 0$ for $n \geq 2$
- $\{\ln(n)\}$ increases, so $\left\{\frac{1}{\ln(n)}\right\}$ decreasing.
- $\lim_{n \to \infty} \frac{1}{\ln(n)} = \frac{1}{\infty} = 0$

So $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges.

**Note:** AST never says “diverges”. 
Now suppose we estimate the sum $S$ by $S_N$. Then

$$|S_N - S| \leq a_{N+1} \leq 10^{-16}$$

$$\frac{1}{\ln(N+1)} \leq 10^{-16}$$

$$N \geq e^{10^{16}} \approx 4.3429 \times 10^{434000000000000000}$$

**Ex. 5**

Determine convergence of each series:

(a) \[ \sum_{n=2}^{\infty} \frac{\ln(n)}{n^3} \]

(b) \[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n} + 1} \]

(c) \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}} \]

(d) \[ \sum_{n=1}^{\infty} 7^{-n} \]

Solution: (Some hints)
(a) Integral Test (integration by parts) or LCT with comparison to $\sum \frac{1}{n^2}$.

(b) AST

(c) LCT with comparison to $\sum \frac{1}{n}$.

(d) Geometric series.

(e) Since $\{\sin(n)\}$ does not alternate in sign, we cannot use AST. Use DCT and absolute convergence test.

\[ \frac{|\sin(n)|}{n^2} \leq \frac{1}{n^2} \]

(f) LCT with comparison to $\sum \frac{1}{n^2}$.

(Intuition: $e^n \geq n^a$ for any $a > 0$, eventually. So $\frac{n}{e^n} \leq \frac{n}{e^n} \leq \frac{n}{n^a} = \frac{1}{n^{a-1}}$. So eventually choose $a > 0$ and $a-1 > 1$, or $a > 2$.)

(g) Nth-term divergence test.
Theorem #1 (Ratio Test)

Suppose \( a_n \neq 0 \) and the following limit exists or is \( \infty \):

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|
\]

"Rho"

(i) \( \rho < 1 \): \( \sum a_n \) converges absolutely

(ii) \( \rho > 1 \): \( \sum a_n \) diverges

(iii) \( \rho = 1 \): test is inconclusive

Proof:

(i) If \( \rho < 1 \), choose \( r \) so \( \rho < r < 1 \).

Since

\[
\left| \frac{a_{n+1}}{a_n} \right| \to \rho,
\]

for all \( n \geq N \) with \( N \) large enough, we have

\[
\left| \frac{a_{n+1}}{a_n} \right| < r,
\]

or

\[
|a_{n+1}| < r |a_n|
\]

Therefore,

\[
|a_{n+1}| < r |a_n|
\]

Therefore, \( \sum a_n \) converges absolutely.
\[ |a_{N+2}| < r |a_{N+1}| < r^2 |a_N| \]
\[ |a_{N+2}| < r \left( |a_{N+2}| < r^3 |a_N| \right) \]

and so on...

\[ |a_{N+n}| < r^n |a_N| \]

Hence...

\[ \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n| \]

Some \# since sum \( |a_{N+k}| < r^k |a_N| \) is finite

\[ = \sum_{n=0}^{N-1} |a_n| + \sum_{k=0}^{\infty} |a_{N+k}| \]

\[ < \sum_{n=0}^{\infty} |a_n| + \sum_{k=0}^{\infty} r^k |a_N| \]

\[ < \sum_{n=0}^{N-1} |a_n| + |a_N| \sum_{k=0}^{\infty} r^k \]

\[ = \sum_{n=0}^{\infty} |a_n| + |a_N| \sum_{k=0}^{\infty} r^k \]

Converges since \( |r| < 1 \)
< \infty \) (series converges)

(ii) Similar to (i).

(iii) Consider the series

\[ \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \]

\[ \sum_{n=1}^{\infty} \frac{1}{n} : \ p = \lim_{n \to \infty} \left| \frac{1}{n+1} \right| = 1 \]

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} : \ p = \lim_{n \to \infty} \left| \frac{1}{(n+1)^2} \right| = 1 \]

So \( p = 1 \) could correspond to either a convergent or divergent series. \( \square \)

**Theorem #2 (Root Test)**

Suppose the following limit exists, or is \( \infty \):

\[ p = \lim_{n \to \infty} \left| a_n \right|^{\frac{1}{n}} \]

Same conclusions as Ratio Test.
Ex. 1

Determine convergence of series

\[ \sum_{n=1}^{\infty} \frac{2^n}{n!} \]

\( (n! = n \cdot (n-1) \cdot (n-2) \cdots (2) \cdot (1). ) \)

Solution:

Tip: Factorials indicate that the Ratio Test is likely a good idea!

We have \( (a_n = \frac{2^n}{n!}) \):

\[
S = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left( \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right)
\]

\[ = \lim_{n \to \infty} \left( \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} \right) = \lim_{n \to \infty} \left( 2 \cdot \frac{1}{n+1} \right) \]

\( (n+1)! = (n+1) \cdot n \cdot (n-1) \cdot (n-2) \cdots (2) \cdot (1) \)

\[ = n! \]

\[ \frac{(n+1)!}{n!} = \frac{(n+1) \cdot n!}{n!} = (n+1) \]
Since $p < 1$, \( \sum_{n=1}^{\infty} \frac{2^n}{n!} \) converges. \( \left( \sum_{n=1}^{\infty} \frac{2^n}{n!} = e^2 - 1 \right) \)

Ex. 2

Determine convergence of series

\[ \sum_{n=1}^{\infty} \frac{n^3}{3^n} \]

Solution:

\[ \frac{n^3}{3^n} > \frac{1}{3^n} \quad \text{(useless)} \]

\[ \frac{n^3}{3^n} < n^3 \quad \text{(useless)} \]

\[ \frac{n^3}{3^n} = n^3 \left( \frac{2}{3} \right)^n \rightarrow 0 \quad \text{(some work)} \]

Let's use Root/Ratio Test instead.
**Ratio Test:** \[ a_n = \frac{n^3}{3^n} \]

\[ p = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left( \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \right) = \lim_{n \to \infty} \left( \frac{1}{3} \cdot \left( \frac{n+1}{n} \right)^3 \right) = \frac{1}{3} \cdot \left( 1 + \frac{1}{n} \right)^3 = \frac{1}{3} \]

So \( p < 1 \), so \( \sum_{i=1}^{\infty} \frac{n^3}{3^n} \) converges.

**Root Test:** \[ a_n = \frac{n^3}{3^n} \]

\[ p = \lim_{n \to \infty} \left| a_n \right|^{1/n} = \lim_{n \to \infty} \left( \frac{n^3}{3^n} \right)^{1/n} = \lim_{n \to \infty} \left( \frac{n^{3/n}}{3} \right) = \lim_{n \to \infty} \left( \frac{(n^{1/n})^3}{3} \right) = \frac{1}{3} \]

From Section 10.1 notes, \( n^{1/n} \to 1 \).

Since \( p < 1 \), \( \sum_{i=1}^{\infty} \frac{n^3}{3^n} \) converges.
C^{1/n} \to 1 \quad (C > 0, \text{ constant})

n^{1/n} \to 1

\left(1 + \frac{x}{n}\right)^n \to e^x \quad (x \text{ constant})

\textbf{Ex. 3}

Determine convergence of series

$$
\sum_{n=1}^{\infty} \left(1 - \frac{2}{n}\right)^{n^2}
$$

\textbf{Solution:}

Use Root Test.

$$
g = \lim_{n \to \infty} \left| an^{1/n} \right| = \lim_{n \to \infty} \left[\left(1 - \frac{2}{n}\right)^{n^2}\right]^{1/n}
$$

$$
= \lim_{n \to \infty} \left(1 - \frac{2}{n}\right)^n = e^{-2}
$$

Since \( e^{-2} < 1 \), \( \sum_{n=1}^{\infty} \left(1 - \frac{2}{n}\right)^{n^2} \) converges.

\textbf{Ex. 4}
Determine convergence of series

\[ \sum_{n=0}^{\infty} \frac{(-1)^n n!}{500^n} \]

**Solution:**

Use Ratio Test.

\[ S = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} (n+1)!}{500^{n+1}} \cdot \frac{500^n}{(-1)^n n!} \right| \]

\[ = \lim_{n \to \infty} \left( \frac{500^n}{500^{n+1}} \cdot \frac{(n+1)!}{n!} \right) = \lim_{n \to \infty} \left( \frac{1}{500} \cdot \frac{(n+1)}{n} \right) \]

\[ = \lim_{n \to \infty} \left( \frac{n+1}{500} \right) = \infty \]

Since \( \infty > 1 \), \[ \sum_{n=0}^{\infty} \frac{(-1)^n n!}{500^n} \] diverges.

**Ex. 5**

Determine convergence of series

\[ \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \]
Solution:
Use Ratio Test.

\[ p = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left( \frac{[(n+1)!]^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} \right) \]

\[ = \lim_{n \to \infty} \left( \frac{(2n)!}{(2n+2)!} \cdot \frac{(n+1)!}{n!} \right)^2 \]

\[ = \lim_{n \to \infty} \left( \frac{1}{(2n+1)(2n+2)} \cdot (n+1)^2 \right) \]

\[ = \lim_{n \to \infty} \left( \frac{n^2 + \ldots}{4n^2 + \ldots} \right) = \frac{1}{4} \]

Since \( p < 1 \), \( \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \) converges.

**Stirling's Approximation**

There is a constant \( C \) such that

\[ \lim_{n \to \infty} \frac{n!}{n^{n+1/2}e^{-n}} = C \]
(Curiously, \( C = \sqrt{2\pi} \).) This is useful because it says that for \( n \) large, \( n! \) can effectively be replaced with \( C n^{n+\frac{1}{2}} e^{-n} \).

**Note:** \((n!)^2 \sim C^2 n^{2n+1} e^{-2n}\)

\((2n)! \sim C (2n)^{2n+\frac{1}{2}} e^{-2n}\)

----

*Root Test or Ratio Test?*

The Root Test is a stronger test than Ratio Test. In other words, if Ratio Test is conclusive, so is Root Test. But not conversely. See next example.

**Ex. 6**

Determine convergence of series

\[
\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \ldots = \sum_{n=0}^{\infty} \frac{1}{2^n + (-1)^n}
\]
Solution:

**Ratio Test:**

\[ s = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{2^n + (-1)^n}{2^{n+1} + (-1)^{n+1}} \]

\[ = \lim_{n \to \infty} \left( 2^n + (-1)^n - 1 - (-1)^{n+1} \right) \]

\[ = \lim_{n \to \infty} \left( 2 \cdot (-1)^n - 1 \right) \]

\[ \text{n odd: } 2 \cdot (-1)^n - 1 = 2^{-2-1} = \frac{1}{8} \]

\[ \text{n even: } 2 \cdot (-1)^n - 1 = 2^{2-1} = 2 \]

So \( s \) does not exist. Hence Ratio Test is inconclusive.

**Root Test:**

\[ s = \lim_{n \to \infty} \left| a_n \right|^{1/n} = \lim_{n \to \infty} \left( \frac{1}{2^n + (-1)^n} \right)^{1/n} \]

\[ = \lim_{n \to \infty} \left( \frac{1}{2^{1+ (-1)^n/n}} \right) = \frac{1}{2^{1+0}} = \frac{1}{2} \]
\[
\lim_{n \to \infty} \frac{(-1)^n}{n} = 0 \quad (\text{Squeeze Theorem})
\]
\[-1 \leq (-1)^n \leq 1 \quad \Rightarrow \quad -\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}
\]
Since \( p < 1 \), the series converges.

**Ex. 7**

Determine convergence of these:

(a) \[\sum_{n=1}^{\infty} \frac{n!}{n^n}\]

(b) \[\sum_{n=1}^{\infty} \sin \left(\frac{10}{n^2}\right)\]

(c) \[\sum_{n=1}^{\infty} \left(\frac{2 + 3n}{1+4n}\right)^n\]

(d) \[\sum_{n=1}^{\infty} \frac{n^2 - 3^n}{(2n+1)!}\]

**Solution:**

(a) Use Ratio Test.

\[
\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \frac{1}{n+1}
\]

\[
= \frac{(n+1)!}{} \cdot \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n
\]

cancel one power}
= \left( \frac{n+1}{n} \right)^{-n} = \left( 1 + \frac{1}{n} \right)^{-n} = \left[ \left( 1 + \frac{1}{n} \right)^n \right]^{-1}

Now we have

\[
p = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left[ \left( 1 + \frac{1}{n} \right)^n \right]^{-1} = e^{-1}
\]

Since \( e^{-1} < 1 \), series converges.

(b) **Hint:** \( \sin(x) \leq x \) for all \( x \geq 0 \)

So this suggests comparison to \( \sum \frac{1}{n^2} \).

To deal with the possibility of negative terms, look at absolute convergence.

Observe that

\[
\lim_{n \to \infty} \frac{|\sin \left( \frac{10}{n^2} \right)|}{\frac{1}{n^2}} = \lim_{u \to 0^+} \left| \frac{\sin (10u)}{u} \right|
\]

\[u = \frac{1}{n^2}, \text{ then } n \to \infty \text{ implies } u \to 0^+\]

\[
= \left| \lim_{u \to 0^+} \frac{\sin (10u)}{u} \right| = |10| = 10
\]

Observe that \( \sum \frac{1}{n^2} \) converges by \( p \)-test
\((p = 2 > 1)\). So by LCT, \(\sum |\sin(10/n^2)|\) converges. Hence \(\sum \sin(10/n^2)\) converges since it converges absolutely.

(c) Use Root Test \((p = 3/4)\), converges.

(d) Use Ratio Test \((p = 0)\), converges.
Section 10.6: Power Series

**Definition #1:**
A power series with center $c$ is an infinite series of the form

$$F(x) = a_0 + a_1 (x-c) + a_2 (x-c)^2 + a_3 (x-c)^3 + \ldots$$

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \quad (\text{convention: } 0^0 = 1)$$

**Theorem #1:**
For any power series $\sum_{n=0}^{\infty} a_n (x-c)^n = f(x)$, exactly one of the following is true:

1. There is a positive number $R$, called the radius of convergence, such that $f(x)$ converges absolutely for $|x-c| < R$ and $f(x)$ diverges for $|x-c| > R$.

2. $f(x)$ converges for all $x$. (radius of convergence is $R = \infty$)
In other words, the convergence of a power series looks like:

\[ \text{divergence} \quad \text{convergence} \quad \text{divergence} \]

anything can happen at \( x = c \pm R \).

Primary goal for a given series is to find the interval of convergence.

**Ex. 1**

Determine ROC (radius of conv.) and IOC (interval of conv.) for the series

\[ f(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n} \]

**Solution:**
Note the center is $x = 0$. Find the ROC using Ratio or Root Test.

**Ratio Test**

\[
\rho = \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{x^{n+1}}{2^n} \cdot \frac{2^n}{2^{n+1}} \right| = \lim_{n \to \infty} \frac{|x|}{2} = \frac{|x|}{2}
\]

So we conclude:

- $\rho < 1$: ($\frac{|x|}{2} < 1$, or $|x| < 2$) convergence
- $\rho > 1$: ($\frac{|x|}{2} > 1$, or $|x| > 2$) divergence
- $\rho = 1$: ($\frac{|x|}{2} = 1$, or $x = \pm 2$) ????

So right now we know that the ROC is $R = 2$ and the IOC is at least as large as $(-2, 2)$, but it may include one or both endpoints. We test the two endpoints separately.
This series diverges by Nth term div. test since \( \lim_{n \to \infty} (-1)^n \neq 0 \).

So our final answer is:

\[
\text{ROC : 2}
\]

\[
\text{IOC : (-2, 2)}
\]
$$p = \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \to \infty} \left| \frac{(-5)^{n+1} (x+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-5)^n (x+1)^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(-5)^{n+1}}{(-5)^n} \cdot \frac{n!}{(n+1)!} \cdot \frac{(x+1)^{n+1}}{(x+1)^n} \right|$$

$$= \lim_{n \to \infty} \left| (-5) \cdot \frac{1}{n+1} \cdot (x+1) \right|$$

$$= \lim_{n \to \infty} \frac{5 |x+1|}{n+1} = 0 \quad x \text{ is fixed and } n \to \infty$$

So we conclude: since $p < 1$ for all values of $x$, the series converges for all $x$.

ROC: $\infty$

IOC: $(-\infty, \infty)$

---

**Ex. 3**

Find ROC and IOC for the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2 \cdot 3^n} (x-2)^n$$

**Solution:**
Use Root Test.

\[ f = \lim_{{n \to \infty}} \left( |b_n|^{1/n} \right) = \lim_{{n \to \infty}} \left| \frac{\sqrt{n^3 + 1}}{n^2 \cdot 3^n} (x-2)^n \right|^{1/n} \]

\[ = \lim_{{n \to \infty}} \frac{1}{2^n} \left( \frac{n^3 + 1}{n^{2/n} \cdot 3} \right) (x-2) \]

\[ = \frac{|x-2|}{3} \lim_{{n \to \infty}} \left( \frac{(n^3 + 1)^{1/n}}{(n^{1/n})^2} \right)^{1/2} = \frac{|x-2|}{3} \]

\[ \lim_{{n \to \infty}} n^{1/n} = 1 \]

\[ \lim_{{n \to \infty}} (n^3 + 1)^{1/n} = 1 \]

Let \( L = \lim_{{n \to \infty}} (n^3 + 1)^{1/n} \). Then

\[ \ln(L) = \lim_{{n \to \infty}} \frac{\ln(n^3 + 1)}{n} = \lim_{{n \to \infty}} \frac{3n^2}{n^3 + 1} = 0 \]

Since \( \ln(L) = 0 \), \( L = 1 \).

So we conclude that:

\[ f < 1: \ (x-2) < 3 \ , \ \text{convergence} \]
\[ p > 1: \quad |x-2| > 3, \quad \text{divergence} \]
\[ p = 1: \quad |x-2| = 3, \quad ?? \]

Test endpoints separately:

\[ x = -1: \quad \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2 \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n^3+1}}{n^2} \]

Put \( a_n = \frac{\sqrt{n^3+1}}{n^2} = \sqrt{\frac{n^3+1}{n^4}} = \sqrt{\frac{1}{n} + \frac{1}{n^4}} \)

- \( a_n \geq 0 \quad (\text{obvious}) \)
- \( \lim_{n \to \infty} a_n = \sqrt{0 + 0} = 0 \)
- \( \{a_n\} \) is decreasing
- \( \frac{1}{n} \) and \( \frac{1}{n^4} \) are decreasing
- \( \frac{1}{n} + \frac{1}{n^4} \) is decreasing
- \( \sqrt{\frac{1}{n} + \frac{1}{n^4}} \) is decreasing

So our series converges by AST.

\[ x = 5: \quad \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2 \cdot 3^n} = \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2} \]

\[ \frac{\sqrt{n^3+1}}{n^2} \sim \frac{\sqrt{n^3}}{n^2} \sim \frac{n^{3/2}}{n^2} \sim \frac{1}{n^{1/2}} \]
\[
\lim_{n \to \infty} \frac{\sqrt{n^3+1}}{n^2} = \lim_{n \to \infty} \sqrt{\frac{n^4+n}{n^4}} = \sqrt{1} = 1
\]

Since \( \frac{1}{\sqrt{n}} \) diverges by \( p \)-test \( (p=\frac{1}{2} \leq 1) \), our series diverges by LCT.

**Final answer for power series:**

**ROC:** \( R = 3 \)

**IOC:** \( [-1, 5) \)

---

We can also use known power series to derive new power series. If \( |x| < 1 \),

\[
1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}
\]

We can use the geometric series to derive other series. In the sequel,

\[
g(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{Converges for } |x| < 1
\]
Ex. 4

Find a power series representation of
\[ f(x) = \frac{1}{1 - 2x} \]
with center \( c = 0 \). Find its ROC and IOC.

**Solution:**
We use the geometric series \( g(x) \).

Observe that
\[ f(x) = g(2x) \]

So then we have...

\[ f(x) = \frac{1}{1 - 2x} = g(2x) \quad g(x) \text{ converges for } |x| < 1 \]

\[ f(x) = \sum_{n=0}^{\infty} (2x)^n \quad \text{converges for } |2x| < 1 \]

\[ f(x) = \sum_{n=0}^{\infty} 2^n x^n = 1 + 2x + 4x^2 + 8x^3 + \ldots \]

We do not need to use Ratio Test
to find the ROC or IOC! Why?

Our series for \( f(x) \) converges for 
\[(2x) < 1, \text{ or } |x| < \frac{1}{2}.\]

center: \( c = 0 \)

ROC: \( R = \frac{1}{2} \)

IOC: \( (-\frac{1}{2}, \frac{1}{2}) \)

Theorem #2:
Consider the power series
\[ f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \]
Suppose \( f \) has an ROC of \( R > 0 \).
(Possibly \( R = \infty \).) Then \( f \) is differentiable and integrable on \((c-R, c+R)\). The derivatives and integrals are obtained by term-by-term operations:

How do the ROC and IOC change by differentiating or integrating?
*In general, the derivative $F'$ and function $F$ have the same RoC. (Same for antiderivatives.)

* Differentiating a power series can only lose you convergence at one or both endpoints.

* Integrating a power series can only gain you convergence at one or both endpoints.

**Ex:** $f(x) = \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2 \cdot 3^n} (x-2)^n$

converges on $[-1, 5)$. Then $f$ is diff. and integrable on $(-1, 5)$.

- $f'(x) = \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2 \cdot 3^n} \frac{d}{dx} (x-2)^n$

- $f'(x) = \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n \cdot 3^n} (x-2)^{n-1}$
Find a power series representation of \( f(x) = \frac{1}{(1-x)^2} \) with center \( c=0 \).

Find the ROC and IOC.

**Solution:**

Recall: \( g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad (|x| < 1) \)

Observe that \( g'(x) = f(x) \)

Note: We also have \( f(x) = g(x)^2 \).

So \( f(x) = \left(1 + x + x^2 + x^3 + \ldots\right)^2 = \ldots \), which is more difficult to compute than \( g'(x) \).

\[ g(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \quad |x| < 1 \]
\[ g'(x) = \sum_{n=0}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \ldots \quad (|x| < 1) \]

* Differentiation can only lose you endpoint convergence, but we don't have endpoint convergence anyway!

So we have

\[ f(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}, \quad (|x| < 1) \]

---

**Ex. 6**

Find a power series representation for \( f(x) = \tan^{-1}(x) \). Find the ROC and IOC.

**Solution:**

Note: \[ \frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2} \] geometric series??
Recall: \( g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \), \( |x| < 1 \)

So we have (replacing \( x \) with \(-x^2\))...

\[
\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = g(-x^2)
\]

\[
\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n \quad \text{converges for } \quad |x| < 1
\]

\[
\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}
\]

So now integrating gives

\[
\int \frac{1}{1+x^2} \, dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx
\]

\[
\tan^{-1}(x) + C = \sum_{n=0}^{\infty} (-1)^n \int x^{2n} \, dx
\]

\[
\tan^{-1}(x) + C = \sum_{n=0}^{\infty} (-1)^n \frac{X^{2n+1}}{2n+1}
\]

\[
\tan^{-1}(x) + C = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots
\]

To find value of \( C \), substitute \( x = 0 \).
\[
\tan^{-1}(0) + C = 0 - 0 + 0 + \ldots \\
\Rightarrow C = 0
\]

Hence we have found
\[
\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}
\]

Where does this series converge?

Since the power series for \(\frac{1}{1+x^2}\) converges for \(-1 < x < 1\), the power series for \(\tan^{-1}(x)\) converges for \(-1 < x < 1\). So we check \(x = -1\) and \(x = 1\).

\[
\begin{align*}
\text{\textcolor{blue}{x = -1}} & \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (-1)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} \\
\text{converges by AST (let } b_n = \frac{1}{2n+1}) \quad & \quad b_n > 0 \quad \text{\textcolor{green}{b_n \rightarrow 0}} \\
\text{\textcolor{blue}{x = 1}} & \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (1)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \\
\text{converges by AST (or it's just}}
\end{align*}
\]
So the IOC of our series for \( \tan^{-1}(x) \) is \([-1, 1]\).

\[
\tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots
\]

\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots
\]

**Ex. 7**

Determine a power series representation of \( f(x) = \frac{1}{8 + x^3} \) with center \( c = 0 \).

Find the ROC and IOC.

**Solution**:

Start with the fact:

\[
g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (|x| < 1)
\]

Observe that
\[
f(x) = \frac{1}{8} \cdot \frac{1}{1 + \frac{x^3}{8}} = \frac{1}{8} \cdot \frac{1}{1 - (-\frac{x^3}{8})}
\]

\[
f(x) = \frac{1}{8} \sum (-\frac{x^3}{8})^n = \frac{1}{8} \sum_{n=0}^{\infty} (-\frac{x^3}{8})^n
\]

\[
f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{8^{n+1}} \quad \text{(where does this converge?)}
\]

Recall that \( g(x) \) converges for \( |x| < 1 \).
So \( f(x) \) converges for \( |-x^3/8| < 1 \), or \( |x| < 2 \). So we have:

\[
\text{ROC: } 2
\]

\[
\text{IOC: } (-2, 2)
\]
Section 10.7: Taylor Series

Given \( f(x) \), how do we generally find a power series representation with center \( x = c \)?

\[
f(x) = a_0 + a_1 (x-c) + a_2 (x-c)^2 + a_3 (x-c)^3 + \ldots
\]

Our question is equivalent to asking how to find \( a_n \) for any \( n \).

\[
\begin{align*}
    f(x) &= a_0 + a_1 (x-c) + a_2 (x-c)^2 + a_3 (x-c)^3 + \ldots \\
    f(c) &= a_0 + 0 + 0 + 0 + \ldots \\
    f'(c) &= a_1 + a_0 \\
    f''(x) &= a_1 + 2a_2 (x-c) + 3a_3 (x-c)^2 + \ldots \\
    f''(c) &= a_1 + 0 + \ldots \\
    f'''(c) &= a_1 \\
    f'''(x) &= 2a_2 + 6a_3 (x-c) + 12a_4 (x-c)^2 + \ldots \\
    f'''(c) &= 2a_2 + 0 + \ldots \\
    f''''(c) &= 2a_2 \\
\end{align*}
\]
So we find that, in general,

\[ a_n = \frac{f^{(n)}(c)}{n!} \quad f^{(n)}(x) = f(x) \quad 0! = 1 \]

(Cauchy formula)

**Definition #1:**
Given a \( N \)-times differentiable function \( f(x) \) at \( x = c \), its \( N \)-degree Taylor polynomial is

\[ T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!} (x-c)^n \]

The Taylor series of \( f \) at \( x = c \) (or about \( x = c \)) is

\[ T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n \]

(assuming \( f \) is infinitely differentiable).

If \( c = 0 \), we call these Maclaurin polynomials and the Maclaurin series.
Some comments:

* In general, even if $T(x)$ converges, it may not converge to $f(x)$.

Answering whether $T(x)$ converges to $f(x)$ is beyond the scope of 152.

* If $f(x)$ does have a convergent power series representation about $x = c$, then it must be given by $T(x)$.

So power series with a given center are unique. You can find them by any valid method.

---

Ex. 1

Find the Maclaurin series of $f(x) = \frac{1}{1 - x}$.

Solution:

We use the Cauchy formula

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$a_n = \frac{f^{(n)}(0)}{n!}$$
<table>
<thead>
<tr>
<th>$n$</th>
<th>$f^{(n)}(x)$</th>
<th>$f^{(n)}(0)$</th>
<th>$a_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{1-x}$</td>
<td>1</td>
<td>$\frac{1}{0!} = 1$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{(1-x)^2}$</td>
<td>1</td>
<td>$\frac{1}{1!} = 1$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2 \cdot 1}{(1-x)^3}$</td>
<td>2</td>
<td>$\frac{2}{2!} = 1$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{3 \cdot 2 \cdot 1}{(1-x)^4}$</td>
<td>6</td>
<td>$\frac{6}{3!} = 1$</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td>N</td>
<td>$\frac{N!}{N!} = 1$</td>
</tr>
</tbody>
</table>

So, $a_n = 1$ for all $n$. So our Maclaurin series is

$$T(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

---

**Ex. 2**

Find the Maclaurin for $f(x) = e^x$. Find
its ROC and IOC.

Solution:

We use the Cauchy formula:

<table>
<thead>
<tr>
<th>n</th>
<th>f^{(n)}(x)</th>
<th>f^{(n)}(0)</th>
<th>a_n = \frac{f^{(n)}(0)}{n!}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>e^x</td>
<td>1</td>
<td>1/0! = 1</td>
</tr>
<tr>
<td>1</td>
<td>e^x</td>
<td>1</td>
<td>1/1! = 1</td>
</tr>
<tr>
<td>2</td>
<td>e^x</td>
<td>1</td>
<td>1/2! = 1/2</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>N</td>
<td>e^x</td>
<td>1</td>
<td>1/N! = 1/N</td>
</tr>
</tbody>
</table>

So the Maclaurin series for e^x is

\[ 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n \]

Now use Ratio Test to find ROC.

\[
p = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \to \infty} \frac{|x|}{n+1} = 0
\]

So ROC is \(-\infty, \infty\) and IOC is \((\infty, \infty)\).
Find Maclaurin series of \( \sin(x) \). Find its ROC and IOC.

**Solution:**

We use the Cauchy formula.

\[
\begin{array}{cccc}
n & f^{(n)}(x) & f^{(n)}(0) & a_n = \frac{f^{(n)}(0)}{n!} \\
0 & \sin(x) & 0 & 0 \\
1 & \cos(x) & 1 & 1 \\
2 & -\sin(x) & 0 & 0 \\
3 & -\cos(x) & -1 & -\frac{1}{3!} \\
4 & \sin(x) & \text{repeats} & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\text{cycles in 4's} & \vdots & \vdots & \vdots \\
\end{array}
\]

So the Maclaurin series for \( \sin(x) \) is

\[
x^1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}
\]
The ROC is \( \infty \) and \( \text{IOC is } (-\infty, \infty) \).

### Table of Common Maclaurin Series

You must know the following:

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \quad |x| < 1
\]

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

\[
\sin(x) = x - \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}
\]

\[
\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}
\]

**Ex. 4**

Find Maclaurin series of \( f(x) = x^2 e^{-3x} \).

**Solution:**

Start with power series for \( e^x \).

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]
\[ e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} x^n \quad (\text{all } x) \]
\[ x^2 e^{-3x} = x^2 \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} x^{n+2} \]
\[ = x^2 - 3x^3 + \frac{9}{2} x^4 - \frac{27}{6} x^5 + \ldots \quad (\text{all } x) \]

**Ex. 5**

Estimate the value of \( \int_0^1 \sin(x^2) \, dx \) with an error less than 10^-6.

**Solution**:

(Fresnel Sine function: \( \int_0^x \sin(t^2) \, dt \))

First, we find a convergent power series whose value is \( \int_0^1 \sin(x^2) \, dx \).

\[ \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} X^{2n+1} \quad (\text{all } x) \]

\[ \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} X^{4n+2} \quad (\text{all } x) \]

\[ \int_0^1 \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 X^{4n+2} \, dx \]
\[ \int_0^1 \sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \cdot \frac{x^{4n+3}}{4n+3} \bigg|_0^1 \]

\[ \int_0^1 \sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \cdot \frac{1}{4n+3} \]

Now we estimate the integral with the partial sum:

\[ \sum_{n=0}^{N} (-1)^n \frac{1}{(2n+1)!} \cdot \frac{1}{4n+3} \]

Find \( N \) so the error is less than \( 10^{-6} \).

(First verify hypotheses of AST.)

Put \( b_n = \frac{1}{(2n+1)! (4n+3)} \)

- \( b_n \geq 0 \)
- \( \{b_n\} \) is decreasing \( \{ \) obvious. \( \}
- \( b_n \rightarrow 0 \)

Now determine the value of \( N \) so that \( b_{N+1} < 10^{-6} \)
\[ \frac{1}{(2N+3)! (4N+7)} < 10^{-6} \]

\begin{align*}
N=0 & : \ b_{N+1} = 0.024 \\
N=1 & : \ b_{N+1} = 0.00076 \\
N=2 & : \ b_{N+1} = 0.000013 \\
N=3 & : \ b_{N+1} = 1.45 \times 10^{-7} \\
\end{align*}

So keep the first four terms (up to \( N=3 \)).

---

**Ex. 6**

Find the Taylor series of \( f(x) = \frac{1}{2 + x} \) about \( x = 3 \).

**Solution:**

We can use the Cauchy formula

\[ a_n = \frac{f^{(n)}(3)}{n!} \]

but we will use the geometric series.
\[ g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (|x| < 1) \]

First change variable to shift center back to 0. Put \( u = x - 3 \). \( \text{Then } x = 3 \) corresponds to \( u = 0 \).

\[ f(x) = \frac{1}{2 + x} = \frac{1}{2 + (u+3)} = \frac{1}{5 + u} = h(u) \]

\[ \text{Taylor series of } \frac{1}{2+x} \text{ about } x = 3 \]

\[ \text{Taylor series of } \frac{1}{5+u} \text{ about } u = 0 \]

Using the geometric series, we have

\[ \frac{1}{5+u} = \frac{1}{5} \cdot \frac{1}{1+\frac{u}{5}} = \frac{1}{5} \cdot \frac{1}{1-\left(-\frac{u}{5}\right)} \]

\[ \text{Find Taylor series about } u = 0 \]

\[ \frac{1}{5+u} = \sum_{n=0}^{\infty} \frac{1}{5} \cdot \left(-\frac{u}{5}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} u^n \]
converges for \( |u/5| < 1 \)
or, for \( |u| < 5 \).
Substituting \( u = x - 3 \), we have

\[
\frac{1}{2 + x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} (x-3)^n
\]

\( \Rightarrow \) converges for \( |x-3| < 5 \)

The ROC is 5 and IOC is \((-2, 8)\).

**Ex. 7**

Use Taylor series to calculate the limit

\[
\lim_{x \to 0} \left( \frac{2\cos(x) - 2 + x^2}{x^4} \right)
\]

(Do not use L'Hôpital's Rule.)

**Solution:**

Expand the numerator in a Taylor series about \( x = 0 \).

\[
2\cos(x) - 2 + x^2 =
\]
\[
2 \left( 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots \right) - 2 + x^2
\]

\[
= 2 - x^2 + \frac{x^4}{12} - \frac{x^6}{360} + \ldots - 2 + x^2
\]

Exact series, but...

If \( x \) is small, first term is good enough approximation.

Now calculate the limit.

\[
\lim_{x \to 0} \left( \frac{2 \cos(x) - 2 + x^2}{x^4} \right) = \frac{x^4}{12} - \frac{x^6}{360} + \ldots
\]

\[
= \lim_{x \to 0} \left( \frac{x^4}{12} - \frac{x^6}{360} + \ldots \right)
\]

\[
= \lim_{x \to 0} \left( \frac{1}{12} - \frac{x^2}{360} + \ldots \right) = \frac{1}{12}
\]
Find the first three nonzero terms of the Taylor series of
\( f(x) = (1-3x)^{3/4} \)
about \( x = -5 \).

**Solution:**

We will use the Cauchy formula:

\[
\begin{align*}
&\ n \\ &\ f^{(n)}(x) \\ &\ f^{(n)}(-5) \\ &\ a_n = \frac{f^{(n)}(-5)}{n!}
\end{align*}
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(-5) )</th>
<th>( a_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>((1-3x)^{3/4})</td>
<td>(16^{3/4} = 8)</td>
<td>(\frac{8}{6!} = \frac{8}{720} = 8)</td>
</tr>
<tr>
<td>1</td>
<td>(-\frac{9}{4}(1-3x)^{-1/4})</td>
<td>(-\frac{9}{4}16^{-1/4} = -\frac{9}{8})</td>
<td>(-\frac{9}{8}/1! = -\frac{9}{8})</td>
</tr>
<tr>
<td>2</td>
<td>(-\frac{27}{16}(1-3x)^{-5/4})</td>
<td>(-\frac{27}{16}16^{-5/4} = \frac{27}{512})</td>
<td>(-\frac{27}{512}/2! = \frac{27}{1024})</td>
</tr>
</tbody>
</table>

The first three nonzero terms are:

\[
T_2(x) = 8 - \frac{9}{8}(x+5) - \frac{27}{1024}(x+5)^2
\]

\[x - c = x - (-5)\]
What is the Taylor series of $f(x) = |x|$?

(a) about $x = 0$?
(b) about $x = 1$?

**Solution:**

(a) Since $|x|$ is not differentiable at $x = 0$, there is no such Taylor series.

(b) If $x > 0$, $|x| = x$. So then $f(x)$ is infinitely differentiable for $x > 0$.

<table>
<thead>
<tr>
<th>n</th>
<th>$f^{(n)}(x)$</th>
<th>$f^{(n)}(1)$</th>
<th>$a_n = \frac{f^{(n)}(1)}{n!}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$</td>
<td>x</td>
<td>= x$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\frac{1}{1!} = 1$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\vdots$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N&gt;1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
So the Taylor series for |x| about x = 1 is

\[ T(x) = 1 + 1 \cdot (x-1) + O(x-1)^2 + \ldots \]

**Ex. 10**

Let \( f(x) = x^4 \cos (3x^2) \). Calculate \( f^{(18)}(0) \) and \( f^{(20)}(0) \).

**Solution:**

Clearly do not calculate 18 derivatives of this function.

\[ \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \]  
\( \text{cos} \) (all x)

\[ \cos(3x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x^2)^{2n}}{(2n)!} \]

\[ = \sum_{n=0}^{\infty} (-1)^n \frac{9^n x^{4n}}{(2n)!} \]  
\( \text{cos} \) (all x)

\[ x^4 \cos(3x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{(2n)!} x^{4n+4} \]  
\( \text{cos} \) (all x)
The Cauchy formula says

\[ f^{(n)}(0) = n! \underbrace{a_n}_{\text{the coefficient of } x^n} \]

For example, for this series...

\[ x^4 \cos(3x^2) = x^4 - \frac{9}{2} x^8 + \frac{81}{4!} x^{12} + \ldots \]

So we have:

\[
\begin{align*}
  a_0 &= a_1 = a_2 = a_3 = 0 \\
  a_4 &= 1 \\
  a_5 &= a_6 = a_7 = 0 \\
  a_8 &= -\frac{9}{2}
\end{align*}
\]

So we have...

\[
\begin{align*}
  f^{(18)}(0) &= 18! \cdot a_{18} = 18! \cdot 0 = 0 \\
  f^{(26)}(0) &= 20! \cdot a_{20} = \frac{20!}{8!} \cdot q^4 \\
  &= (-1)^4 \frac{q^4}{8!}
\end{align*}
\]
Find the Maclaurin series of

\[ f(x) = \frac{x^{18}}{(1+x)^3} \]

Solution:
We will not use Cauchy formula!

\[ g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (|x| < 1) \]

Find series of \( \frac{1}{(1+x)^3} \) first.

\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x| < 1 \]

\[ \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad |x| < 1 \]

\[ \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^{n-2} \quad |x| < 1 \]

\[ \frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2} \quad (|x| < 1) \]
\[
\frac{1}{(1+x)^3} = \sum_{n=2}^{\infty} \frac{(-1)^n n(n-1)}{2} x^{n-2} \quad |x| < 1
\]

\[
\frac{x^{18}}{(1+x)^3} = \sum_{n=2}^{\infty} \frac{(-1)^n n(n-1)}{2} x^{n+6} \quad |x| < 1
\]
Section 7.9: Numerical Integration

* Many integrals cannot be computed by antiderivative

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \]  

**probability**

\[ \text{Si}(x) = \int_0^x \sin(t^2) dt \]  

**optics**

\[ \text{Li}(x) = \int_2^x \frac{1}{\ln(t)} dt \]  

**number theory**

We will look at three methods to estimate \( \int_a^b f(x) \, dx \)

for continuous \( f \). For each method, the interval \([a, b]\) is divided into \( N \) subintervals of equal width.

width of each subinterval: \( \Delta x = \frac{b-a}{N} \)
Midpoint Rule

\[ M_N = \text{total area of rectangles} \]

\[ M_N = \Delta x \left( f(c_1) + f(c_2) + \ldots + f(c_N) \right) \]

where \( c_k \) is the midpoint of \([x_{k-1}, x_k]\).

\[ c_k = a + (k-\frac{1}{2})\Delta x \quad (k=1, 2, \ldots N) \]

Alternative interpretation of \( M_N \):

midpoint rectangle
tangential trapezoid
The area of the tangent trap. and midpoint rect. are equal. So midpoint rule offers higher accuracy than left or right endpoint rule.

**Trapezoid Rule**

Each trapezoid has the following form:

\[ y_{k-1} = f(x_{k-1}) \quad y_k = f(x_k) \]

Area of trapezoid \( k \) = \( \Delta x \left( \frac{y_{k-1} + y_k}{2} \right) \)
So the total area of the trapezoids is:

\[ T_N = \Delta x \left( \frac{y_0 + y_1}{2} \right) + \Delta x \left( \frac{y_1 + y_2}{2} \right) + \Delta x \left( \frac{y_2 + y_3}{2} \right) + \ldots \]

\[ + \ldots + \Delta x \left( \frac{y_{N-1} + y_N}{2} \right) \]

\[ T_N = \frac{1}{2} \Delta x \left( y_0 + 2y_1 + 2y_2 + \ldots + 2y_{N-1} + y_N \right) \]

**Alternative interpretation of** $T_N$:

The idea is to approximate $f(x)$ with Simpson's Rule:

\[ T_N = \frac{L_N + R_N}{2} \]
Using parabolas:

Parallel, vertical, flat sides with parabolic top.

*Note: Three points determine a parabola.

Simpson's Rule requires $N$ to be even.

Area under parabola \[ = \frac{\Delta X}{3} \left( y_{k-1} + 4y_k + y_{k+1} \right) \]

Proof of this formula requires some non-trivial algebra.

So the total area under all parabolas:

\[ S_N = \frac{\Delta X}{3} \left( y_0 + 4y_1 + y_2 \right) + \frac{\Delta X}{3} \left( y_2 + 4y_3 + y_4 \right) + \]
\[ + \frac{\Delta X}{3} \left( y_4 + 4y_5 + y_6 \right) + \cdots + \]
\[ + \frac{\Delta X}{3} \left( y_{N-2} + 4y_{N-1} + y_N \right) \]
\[ S_N = \frac{\Delta x}{3} \left( y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 4y_{n-1} + y_n \right) \]

Coefficient pattern: \( 1, 4, 2, 4, 2, \ldots, 2, 4, 2, 4, 1 \)

Curiously,

\[ S_N = \frac{2}{3} M_{N/2} + \frac{1}{3} T_{N/2} \]

---

**Theorem #1 (Error for \( M_N, T_N \))**

Suppose \( f'' \) is continuous and \( K_2 \) is a number such that

\[ |f''(x)| \leq K_2 \]

for all \( x \) in \([a,b]\). Then the error for \( M_N \) and \( T_N \) satisfy:

\[ \left| \int_a^b f(x) \, dx - T_N \right| \leq \frac{K_2 \,(b-a)^3}{12 \, N^2} = \frac{K_2 \,(b-a)}{12} \cdot (\Delta x)^2 \]

\[ \left| \int_a^b f(x) \, dx - M_N \right| \leq \frac{K_2 \,(b-a)^3}{24 \, N^2} = \frac{K_2 \,(b-a)}{24} \cdot (\Delta x)^2 \]
Theorem #2 (Error for $S_n$)
Suppose $f^{(n)}$ is continuous and $K_N$ is a number such that

$$|f^{(n)}(x)| \leq K_N$$

for all $x$ in $[a, b]$. Then the error of $S_n$ satisfies:

$$\left| \int_a^b f(x) \, dx - S_n \right| \leq \frac{K_N (b-a)^5}{(80 \, N^4)} = \frac{K_N (b-a)^5}{80} \cdot (\Delta x)^4$$

---

Ex. 1

Consider $\int_1^3 \sin(x^2) \, dx$.

(a) Calculate $T_8$.

(b) Use computer for $T_{50}$, $T_{100}$, $T_{500}$, $T_{1000}$, $T_{10000}$.

Solution:

(a) First calculate $\Delta x$. 
\[ \Delta x = \frac{3 - 1}{8} = \frac{2}{8} = \frac{1}{4} = 0.25 \]

Now use formula for \( T_n \).

\[ T_8 = \frac{\Delta x}{2} (y_0 + 2y_1 + 2y_2 + \ldots + 2y_7 + y_8) \]

\[ = \frac{\Delta x}{2} \left( f(x_0) + 2f(x_1) + \ldots + 2f(x_7) + f(x_8) \right) \]

\[ = \frac{\Delta x}{2} \left( f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + 2f(2) + 2f(2.25) + 2f(2.5) + 2f(2.75) + f(3) \right) \]

Now use \( f(x) = \sin(x^2) \).

\[ T_8 = \frac{0.25}{2} \left( \sin(1^2) + 2 \sin(1.25^2) + \ldots + 2 \sin(2.75^2) + \sin(3^2) \right) \]

\[ = 0.4281 \]

(b) In general, \( T_n \) is given by:
\[ T_N = \frac{2}{N} \cdot \frac{1}{2} \left( \sin(1^2) + 2 \sum_{k=1}^{N-1} \sin \left( (1 + \frac{2k}{N})^2 \right) + \sin(3^2) \right) \]

\[ T_{500} = 0.4632855 \]

\[ T_{1000} = 0.4632920 \]

\[ T_{10000} = 0.4632942 \]

\[ \text{not very good!} \]

---

**Ex. 2**

Consider \( \int_1^4 \ln(x) \, dx \).

(a) Calculate \( N_6 \) and \( T_6 \).

(b) Calculate error bounds in theorem.

(c) Verify the theorem.

**Solution:**

(a) \( \Delta x = \frac{4 - 1}{6} = \frac{3}{6} = 0.5 \)

\[ T_6 = \frac{\Delta x}{2} \left( y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + y_6 \right) \]

\[ = 0.25 \left( \ln(1) + 2 \ln(1.5) + 2 \ln(2) + \ldots + 2 \ln(3.5) + \ln(4) \right) \]
\[ T_N \text{ has } N+1 \text{ terms} \]

\[ M_6 = \Delta x \left( f(c_1) + f(c_2) + \ldots + f(c_6) \right) \]

\[ = 0.5 \left( \ln(1.25) + \ln(1.75) + \ln(2.25) + \ldots + \ln(3.75) \right) \]

\[ \approx 2.55285 \quad M_N \text{ has } N \text{ terms} \]

(b) For \( M_6 \) and \( T_6 \) we need \( K_2 \).

\[ f(x) = \ln(x) \implies f''(x) = -\frac{1}{x^2} \]

The number \( K_2 \) must satisfy

\[ |f''(x)| \leq K_2 \]

for all \( x \) in \([1, 4]\).

The smallest choice of \( K_2 \) that works is \( K_2 = 1 \).
Now use the error formulas.

\[
\text{error (T}_6) \leq \frac{k_2 (4-1)^3}{12 \cdot 6^2} = \frac{1}{16} = 0.0625
\]

\[
\text{error (M}_6) \leq \frac{k_2 (4-1)^3}{24 \cdot 6^2} = \frac{1}{32} = 0.03125
\]

(c) The exact value of the integral is

\[
\int_1^4 \ln(x) \, dx = 4\ln(4) - 3 = 2.54518
\]

So the errors are:

\[
T_6: \left| 2.54518 - 2.52971 \right| \approx 0.0155
\]

(which is less than 0.0625)

\[
M_6: \left| 2.54518 - 2.55285 \right| \approx 0.00767
\]

(which is less than 0.03125)

Ex. 3

Find N so that \( S_N \) approximates

\[
\int_1^2 \frac{1}{x} \, dx = \ln(2)
\]
within an error of $10^{-6}$.

Solution:
First we need to find the smallest value of $K_4$.

\[ f(x) = \frac{1}{x} \quad f^{(3)}(x) = \frac{-6}{x^4} \]
\[ f'(x) = -\frac{1}{x^2} \quad f^{(4)}(x) = \frac{24}{x^5} \]
\[ f''(x) = \frac{2}{x^3} \]

We need $K_4$ so that

\[ |f^{(4)}(x)| \leq K_4 \]

for all $x$ in $[1, 2]$.

From Theorem #2,

\[ K_4 = 24 \]
\[
\text{error } (S_N) \leq K_4 \frac{(2-1)^5}{180 \cdot N^4} \leq 10^{-6}
\]

\[\implies N \geq \left( \frac{24 \cdot 10^6}{180} \right)^{\frac{1}{4}} \approx 19.11\]

So we need \( N \geq 20 \) (\( N \) must be even)
In general, we think of a curve or part of a curve as the path of a particle. The coordinates are each functions of a third variable, called a parameter.

\[ x = x(t) \quad t: \text{ parameter (time)} \]

\[ y = y(t) \]

\[ P(t) = (x(t), y(t)) \]

\[ P(0): \text{ starting point} \]

\[ P(1): \text{ terminal point} \]

The parameter may have other interpretations (energy, angle, etc.)
Suppose $x(t) = 2t - 4$ and $y(t) = 3 + t^2$. Sketch the curve and indicate the particle's direction. Assume $t \geq 0$.

**Solution:**

We can find an equation for the curve by eliminating the parameter.

$$x = 2t - 4 \implies t = \frac{x + 4}{2}$$

$$y = 3 + t^2 \implies y = 3 + \left(\frac{x + 4}{2}\right)^2$$

Particle travels along the parabola

$$y = 3 + \frac{1}{4}(x+4)^2$$

Starting point

$$y = 3 + \frac{1}{4}(x+4)^2$$
At $t=0$, $(x, y) = (-4, 3)$

As $t$ increases, $x = 2t - 4$ increases. So particle moves to the right.

Ex. 2

Same as Ex. 1, except

$$x(t) = t, \quad y(t) = 3 + \frac{1}{4} (t+4)^2$$

Solution:

We see that

$$y = 3 + \frac{1}{4} (x+4)^2$$

So particle travels along the same parabola as in Ex. 1. What is different?

![Graph showing the trajectory of the particle with the equation $y = 3 + \frac{1}{4} (x+4)^2$.]
At $t=0$, $(x, y) = (0, 7)$.
As $t$ increases, $x = t$ increases. So particle travels to the right, but it has a different starting point.

More examples on curves vs. paths

Consider the following parametrizations, each of which satisfies $y = x^2$. So in each case, the particle travels on the same parabola, but in different ways. $(-\infty < t < \infty)$

1. $x(t) = t^3$
   $y(t) = t^8$

2. $x(t) = t^2$
   $y(t) = t^4$

3. $x(t) = \cos(t)$
   $y(t) = \cos(t)^2$

Tangent lines to parametrized paths
Suppose \( x(t) = t^2 + 1 \) and \( y(t) = t^3 - 4t \).

(a) Find tangent line to path when \( t = 3 \).

(b) Find points on path where tangent line is horizontal.

**Solution:**

(a) Point: \( P = (x(3), y(3)) = (10, 15) \)

Slope: \( m = \left. \frac{dy}{dx} \right|_{t=3} = \left. \frac{dy}{dt} \right|_{t=3} \cdot \left. \frac{dx}{dt} \right|_{t=3} \)

\[
\left. \frac{dy}{dx} \right|_{t=3} = \frac{3t^2 - 4}{2t} \bigg|_{t=3} = \frac{23}{6}
\]

So the tangent line is:

\[
y - 15 = \frac{23}{6} (x - 10)
\]
(b) Set $\frac{dy}{dx} = 0$. This means set $\frac{dy}{dt} = 0$ and verify $\frac{dx}{dt} \neq 0$.

$$\frac{dy}{dt} = 3t^2 - 4 = 0 \Rightarrow t = \pm \frac{2}{\sqrt{3}}$$

$$\left. \frac{dx}{dt} \right|_{t=\pm \frac{2}{\sqrt{3}}} = (2t)\left|_{t=\pm \frac{2}{\sqrt{3}}} \right. \neq 0$$

So path has horizontal tangent lines at $t = \frac{2}{\sqrt{3}}$ and $t = -\frac{2}{\sqrt{3}}$. Those points on the graph are:

$$P_1 = \left( \frac{7}{3}, \frac{16}{\sqrt{27}} \right) \text{ and } P_2 = \left( \frac{7}{3}, -\frac{16}{\sqrt{27}} \right)$$

**Parametrizations of Common Paths**

Often you are given a curve described by an equation with $x$ and $y$, and you must find a suitable parametrization, which means you find $(x(t), y(t))$.
and the domain of $t$. Some examples:

- **Lines:**
  - Vertical line, passing through $(a, b)$.
    \[
    x(t) = a \quad t \in (-\infty, \infty) \\
    y(t) = t
    \]
  - Line through $(a, b)$ with slope $m$. (Particle starts at $(a, b)$ at $t=0$.)
    \[
    x(t) = a + t \quad t \in (-\infty, \infty) \\
    y(t) = b + mt
    \]

  Check: if $t=0$, $(x, y) = (a, b)$. We have $x = a + t$, or $t = x - a$, so $y = b + mt = b + m(x - a)$.

- **Functions**
  - $y = f(x)$ where $x \in D$ (domain)
    \[
    x(t) = t \quad \{ \quad t \in D \\
    y(t) = f(t)
    \]
• $x = g(y)$, where \( y \in D \) (domain)

\[
X(t) = g(t) \quad \forall \ t \in D \\
y(t) = t
\]

• Circles

\[
(x-a)^2 + (y-b)^2 = R^2
\]

circle with center \((a, b)\) and radius \(R > 0\).

Hint: What functions do you know satisfy something that looks like

\[
\begin{align*}
\left( \cos(t) \right)^2 + \left( \sin(t) \right)^2 &= \text{const.}^2 \\
\sim \cos(t) & \sim \sin(t)
\end{align*}
\]

\[
(x-a)^2 + (y-b)^2 = R^2 \\
\left( \frac{x-a}{R} \right)^2 + \left( \frac{y-b}{R} \right)^2 = 1
\]

cos(t) & sin(t)

So our parametrization of a circle is:
\[
\frac{x-a}{R} = \cos(t) \implies x(t) = a + R \cos(t)
\]
\[
\frac{y-b}{R} = \sin(t) \implies y(t) = b + R \sin(t)
\]
\[ t \in [0, 2\pi) \]

Domain of \([0, 2\pi)\) to get one lap. Since \(\frac{dy}{dt}\bigg|_{t=0}\), particle initially travels upward, so particle travels anticlockwise.

Ex. 4

Parametrize the ellipse
\[ 4x^2 + 9y^2 = 36 \]
Solution:
The equation can be written as:

\[ \frac{4x^2}{36} + \frac{9y^2}{36} = 1 \]

\[ \left( \frac{x}{3} \right)^2 + \left( \frac{y}{2} \right)^2 = 1 \]

\( \cos(t) \quad \sin(t) \)

to our parametrization is:

\[ \frac{x}{3} = \cos(t) \implies x(t) = 3\cos(t) \]

\[ \frac{y}{2} = \sin(t) \implies y(t) = 2\sin(t) \]

\( t \in [0, 2\pi) \)

Arc Length of Parametrized Paths

Suppose a particle travels on the path parametrized by

\[ x = x(t), \quad y = y(t). \]
The speed of the particle is:

\[
v(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}
\]

\[
v(t) = \sqrt{x'^2 + y'^2}
\]

The total distance (arc length) traveled by particle from \( t = a \) to \( t = b \) is:

\[
s = \int_{a}^{b} v(t) \, dt = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

**Ex. 5**

Find length of one petal of the cycloid:

\[
x(t) = 2 \left( t - \sin(t) \right)
\]

\[
y(t) = 2 \left( 1 - \cos(t) \right)
\]

(Assume \( 0 \leq t \leq 2\pi \).)

**Solution:**

Use the arc-length formula:

\[
x = 2 \left( t - \sin(t) \right) = 2t - 2\sin(t)
\]
\[ y = 2 \left( 1 - \cos(t) \right) = 2 - 2 \cos(t) \]
\[ \frac{dx}{dt} = 2 - 2 \cos(t) \]
\[ \frac{dy}{dt} = 2 \sin(t) \]
\[ (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 4 \left( 1 - 2 \cos t + \cos^2 t \right) + 4 \sin^2 t \]
\[ = 4 - 8 \cos(t) + 4 \]
\[ = 8 \left( 1 - \cos(t) \right) \]
\[ \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{8 \left( 1 - \cos(t) \right)} \]

**half-angle formula:** \( \sin(\theta)^2 = \frac{1}{2} - \frac{1}{2} \cos(2\theta) \)

\[ = \sqrt{16 \sin (\frac{t}{2})^2} = 4 \left| \sin \left( \frac{t}{2} \right) \right| = 4 \sin \left( \frac{t}{2} \right) \]

If \( 0 \leq t \leq 2\pi \), \( \sin \left( \frac{t}{2} \right) \geq 0 \)

\[ S = \int_0^{2\pi} 4 \sin \left( \frac{t}{2} \right) dt = 16 \]

Ex. 6
Calculate the circumference of a circle of radius $R$ using the arc length formula.

**Solution:**

A parametrization is:

$$
\begin{align*}
  x &= R \cos(t) \quad 0 \leq t < 2\pi \\
  y &= R \sin(t)
\end{align*}
$$

\[
\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{R^2 \sin^2(t) + R^2 \cos^2(t)} = R
\]

$$
\begin{align*}
  s &= \int_0^{2\pi} R \, dt = 2\pi R \quad \text{As expected!}
\end{align*}
$$

---

**Surface Area**

Assume the following:

1. $y(t) \geq 0$.
2. $\frac{dx}{dt} > 0$ for all $t$.

Let $S$ be the surface obtained
by rotating the curve \((x(t), y(t))\) about the \(x\)-axis from \(t=a\) to \(t=b\). Then the surface area of \(S\) is

\[
A = 2\pi \int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

---

**Ex. 6**

Find area of surface obtained by rotating \(c(t) = (\sin(t)^2, \cos(t)^2)\) about \(x\)-axis for \(0 \leq t \leq \pi/2\).

**Solution:**

Let \(x(t) = \sin(t)^2\) and \(y(t) = \cos(t)^2\). Observe \(y(t) > 0\) and \(\frac{dx}{dt} > 0\).

\[
\frac{dx}{dt} = 2\sin(t) \cos(t) = x
\]
\[
\frac{dy}{dt} = -2 \cos(t) \sin(t) = y
\]
\[
\dot{x}^2 + \dot{y}^2 = 4\sin(t)^2 \cos(t)^2 + 4\sin(t)^2 \cos(t)^2 \\
= 8\sin(t)^2 \cos(t)^2 \\
\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{8}\left|\sin(t) \cos(t)\right| \\
= \sqrt{8}\sin(t) \cos(t) \quad 0 \leq t \leq \frac{\pi}{2},
\]
so both \(\sin(t)\) and \(\cos(t)\) are positive.

\[
A = 2\pi \int_0^{\pi/2} \cos(t)^2 \cdot \sqrt{8}\sin(t) \cos(t) \, dt \\
= 2\sqrt{8}\pi \int_0^{\pi/2} \cos(t)^3 \sin(t) \, dt \\
u = \cos(t), \quad -du = \sin(t) \, dt \\
= 2\sqrt{8}\pi \int_1^0 (-u^3) \, du = 2\sqrt{8}\pi \int_0^1 u^3 \, du \\
= \frac{2\sqrt{8}\pi}{4} u^4 \bigg|_0^1 = \frac{\sqrt{8}\pi}{2} = \sqrt{2}\pi
\]
The \((x,y)\)-coordinates of a point are the Cartesian or rectangular coordinates of that point. But \((x,y)\) is just a label. Someone else might assign another label. What if someone uses a rotated set of axes?

The same point \(P\) may have different labels in different coordinate systems.

* You may have seen this in physics already. Common method of solving
an inclined plane problem is to orient your axes so the x-axis is parallel to the plane.

The polar coordinate system assigns different labels to the point \((x,y)\). We use the coordinates:

- \(r\): distance from the origin
- \(\theta\): angle from positive x-axis.

\[
\begin{align*}
x &= r \cos(\theta) \\
y &= r \sin(\theta)
\end{align*}
\]

\[
\begin{align*}
r &= \sqrt{x^2 + y^2} \\
\tan(\theta) &= \frac{y}{x}
\end{align*}
\]

Careful when solving for \(\theta\)!
What do the “grid lines” in each coordinate system look like?

![Grid lines](image)

- Constant $x$
- Constant $y$

![Polar grid lines](image)

- Constant $r$
- Constant $\theta$ (rays)

---

**Ex. 1**

Convert the following points.

(a) $P = (3, 2)$ from rect. to polar.
(b) $P = (-5, 3)$ from rect. to polar.
(c) $P = (3, \frac{\pi}{6})$ from polar to rect.

**Solution**:

(a) $P = (3, 2)$

\[ r = \sqrt{3^2 + 2^2} = \sqrt{13} \]
\[ \tan(\theta) = \frac{2}{3} \]
\[ \theta = \tan^{-1}\left(\frac{2}{3}\right) \]
(b) 

\[ r = \sqrt{5^2 + 3^2} = \sqrt{34} \]

\[ \tan(\theta) = \frac{3}{-5} \]

\[ \theta \neq \tan^{-1}\left(-\frac{3}{5}\right) \]

This is an angle in \([\frac{-\pi}{2}, 0]\).

Better: find reference angle first

\[ \beta = \text{reference angle} \]

\[ \beta = \tan^{-1}\left(\frac{3}{5}\right) \]

\[ \theta = \pi - \beta = \pi - \tan^{-1}\left(\frac{3}{5}\right) \]

(c) 

\[ x = r \cos(\theta) = 3 \cos\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2} \]

\[ y = r \sin(\theta) = 3 \sin\left(\frac{\pi}{6}\right) = \frac{3}{2} \]
Remarks

1. Since cosine and sine are 2π-periodic, the θ-value is not unique. So the two ordered pairs
   \[(r, \theta)\] and \[(r, \theta + 2n\pi)\] \(n \in \mathbb{Z}\)
   represent the same point.

2. The origin has no well-defined angular coordinate. By convention the ordered pair \((0, \theta)\) represents the origin for any \(\theta\).

3. We do allow negative value of \(r\)... By definition, the ordered pair \((-r, \theta)\) is the reflection of \((r, \theta)\) through the origin. So...

   \((-r, \theta) \iff (r, \theta + \pi)\)
Ex. 2

Find two polar labels of \( P = (-1, 1) \) (given in rect.), one with \( r > 0 \) and one with \( r < 0 \).

**Solution:**

\[ r = \sqrt{1^2 + 1^2} = \sqrt{2} \]

\[ \tan(\theta) = \frac{1}{-1} \implies \theta = \frac{3\pi}{4} \quad \text{second quadrant!} \]
(a) 

\[(r, \theta) = (\sqrt{2}, \frac{3\pi}{4})\]

(b) 

\[(r, \theta) = (-\sqrt{2}, -\frac{\pi}{4})\]

Variable \(r\) as a function of \(\theta\)

What does it mean to integrate a function \(r = f(\theta)\)?
Q: What does the region described by the following inequalities look like?

\[ 0 \leq r \leq f(\theta) \]
\[ \alpha \leq \theta \leq \beta \]

“radially simple”

A: See graph.

How do you find the area of such a region?

* Course theme: divide into smaller, easier pieces, examine one piece, then add all the pieces to get a Riemann sum.

Divide the region into circular wedges.
\[ r_j = f(\theta_j) \]

\[ A_j = \frac{1}{2} f(\theta_j)^2 \Delta \theta \]

(area of sector : \( \frac{1}{2} r^2 \theta \))

Adding the individual areas gives

\[ A_{\text{total}} \approx \sum_{j=1}^{N} \frac{1}{2} f(\theta_j)^2 \Delta \theta \]

Now taking \( N \to \infty \), we get

\[ A = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 \, d\theta \]
Find area enclosed by the right half of the curve \( r = 4 \sin(\theta) \).

**Solution:**

First graph the curve. Let's convert to rectangular coordinates.

\[
\begin{align*}
&x = r \cos(\theta) \\
&y = r \sin(\theta)
\end{align*}
\]

\[
\begin{align*}
r &= 4 \sin(\theta) \\
r &= 4 \left( \frac{y}{r} \right) \\
r^2 &= 4y \\
x^2 + y^2 &= 4y \\
x^2 + y^2 - 4y &= 0 \\
x^2 + y^2 - 4y + 4 &= 4 \\
x^2 + (y - 2)^2 &= 4
\end{align*}
\]

So the curve is a circle with center \((0, 2)\) and radius 2.
This region can be described as

\[ 0 \leq \theta \leq \frac{\pi}{2} \]
\[ 0 \leq r \leq 4 \sin(\theta) \]

So the area of the region is

\[
A = \int_{0}^{\frac{\pi}{2}} \frac{1}{2} (4 \sin \theta)^2 \, d\theta = \int_{0}^{\frac{\pi}{2}} 8 \sin^2(\theta) \, d\theta
\]

\[
= \int_{0}^{\frac{\pi}{2}} 4 (1 - \cos(2\theta)) \, d\theta = 2\pi
\]

**Ex. 4**

Find area enclosed by one petal of the rose \( r = \sin(3\theta) \).

**Solution:**

First graph \( r = \sin(3\theta) \). Converting to Cartesian coordinates is very difficult.
We will examine the graph of $r = \sin (3\theta)$ in the $\Theta r$-plane, then use it to graph $r = \sin (3\theta)$ in the $xy$-plane.
Now calculate the area in one petal

\[\gamma = \sin(3\theta)\]

\[A_{\text{one petal}} = \frac{1}{2} \int_0^{\pi/3} \sin^2(3\theta) \, d\theta\]

double angle: \(\sin^2(\beta) = \frac{1}{2} - \frac{1}{2} \cos(2\beta)\)

\[= \frac{1}{4} \int_0^{\pi/3} (1 - \cos(6\theta)) \, d\theta\]

\[= \frac{1}{4} \left( \theta - \frac{1}{6} \sin(6\theta) \right) \bigg|_0^{\pi/3}\]

\[= \frac{1}{4} \left( \frac{\pi}{3} - 0 \right) - \frac{1}{4} \left( 0 - 0 \right) = \frac{\pi}{12}\]
Find area outside the circle $r = 1$ and inside the circle $r = 4 \cos(\theta)$.

Solution:

First graph the equations.

\[ r = 4 \cos(\theta) \quad \cos(\theta) = \frac{x}{r} \]

\[ r = 4 \left( \frac{x}{r} \right) \]
Find the points of intersection.

\[ 4 \cos (\beta) = 1 \implies \beta = \cos^{-1}\left(\frac{1}{4}\right) \]

Now set up integral.

\[ f_{\text{outer}} (\theta) = 4 \cos (\theta) \]
\[
f_{\text{inner}}(\theta) = 1
\]

\[
A = \frac{1}{2} \int_{0}^{\beta} \left( 16 \cos(\theta)^2 - 1 \right) d\theta
\]

\[
= \frac{1}{2} \cdot 2 \cdot \int_{0}^{\beta} \left( 16 \cdot \frac{1}{2} (1 + \cos(2\theta)) - 1 \right) d\theta
\]

\[
\text{symmetry} \quad \text{double-angle}
\]

\[
= \int_{0}^{\beta} \left( 7 + 8 \cos(2\theta) \right) d\theta
\]

\[
= \left[ 7\theta + 4 \sin(2\theta) \right]_{0}^{\beta} = 7\beta + 4\sin(2\beta)
\]

\[
= 7\beta + 8 \sin(\beta)\cos(\beta)
\]

\[
\text{sin}(2\beta) = 2\sin(\beta)\cos(\beta)
\]

\[
= 7\cos^{-1}\left( \frac{1}{4} \right) + 8 \sqrt{\frac{15}{4}} \cdot \frac{1}{4}
\]

Ex. 6

Find area enclosed by lemniscate

\[
r^2 = \cos(2\theta). 
\]
Solution:
Cartesian coordinates probably too hard

\[
\begin{align*}
   r^2 &= \cos(2\theta) \\
   r^2 &= \cos(\theta)^2 - \sin(\theta)^2 \\
   r^2 &= \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2 \\
   r^4 &= x^2 - y^2 \\
   (x^2 + y^2)^2 &= x^2 - y^2
\end{align*}
\]

We will graph this using the method of Ex. 4.

Now graph the equation in the xy-plane piece by piece.
A_{\text{total}} = 2 \cdot \left( \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos(2\theta) \, d\theta \right)

Now find area enclosed by curve

\[ r = \sqrt{\cos(2\theta)} \]

\[ \theta \text{ in } (\frac{\pi}{4}, \frac{3\pi}{4}) \cup (\frac{3\pi}{4}, \frac{5\pi}{4}) \]

\[ \cos(2\theta) < 0 \text{ for } \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4} \]

\[ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4} \]

\[ \frac{3\pi}{4}, \frac{5\pi}{4} \]

\[ \frac{\pi}{4}, \frac{3\pi}{4} \]

\[ \frac{\pi}{4}, \frac{3\pi}{4} \]

\[ \frac{\pi}{4}, \frac{3\pi}{4} \]
The curve \( r = f(\theta) \) has the following parametrization:

\[
\begin{align*}
  x(\theta) &= f(\theta) \cos(\theta) \\
  y(\theta) &= f(\theta) \sin(\theta)
\end{align*}
\]

So the arc length differential \( ds \) is:

\[
ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \ d\theta
\]

After some magic:

\[
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = f(\theta)^2 + \left(f'(\theta)\right)^2
\]

So arc length formula for polar:
Consider the cardioid $r = 1 - \cos(\theta)$.

(a) Graph in $xy$-plane.

(b) Find the total length.

(c) Find the total enclosed area.

Solution:

(a) $y = \sin(\theta)$, $y = \sin(2\theta)$, $y = \sin(3\theta)$, $y = \sin(4\theta)$, $y = \sin(5\theta)$, $y = \sin(6\theta)$.
(b) Use the arc length formula.

\[ s = \int_{0}^{2\pi} \sqrt{f(\theta)^2 + f'(\theta)^2} \, d\theta \]

\[ = \int_{0}^{2\pi} \sqrt{(1 - \cos(\theta))^2 + (\sin(\theta))^2} \, d\theta \]

\[ = \int_{0}^{2\pi} \sqrt{1 - 2\cos(\theta) + \cos^2(\theta) + \sin^2(\theta)} \, d\theta \]

\[ = \int_{0}^{2\pi} \sqrt{2(1 - \cos(\theta))} \, d\theta \]

\[ = \int_{0}^{2\pi} \sqrt{2} \cdot \sqrt{2 \sin^2(\theta/2)} \, d\theta \]

\[ = \int_{0}^{2\pi} 2 \sin(\theta/2) \, d\theta = 8 \]

(c) \[ A = \int_{0}^{2\pi} \frac{1}{2} (1 - \cos(\theta))^2 \, d\theta \]

\[ = \int_{0}^{2\pi} \frac{1}{2} (-2\cos(\theta) + \cos^2(\theta)) \, d\theta \]
\[
= \int_{0}^{2\pi} \frac{1}{2} \left( 1 - 2\cos(\theta) + \frac{1}{2} + \frac{1}{2} \cos(2\theta) \right) d\theta \\
= \int_{0}^{2\pi} \left( \frac{3}{4} - \cos(\theta) + \frac{1}{4} \cos(2\theta) \right) d\theta \\
= 0 \quad = 0 \\
= \frac{3}{4} \cdot 2\pi = \frac{3\pi}{2}
\]
Appendix: Complex Numbers
What should you know from Chapter 1?

- Suppose \( z = x + iy \) and \( w = u + iv \)
  - \( z + w \)
  - \( z - w \)
  - \( 2z, 3w \)
  - \( zw \)
  - \( \frac{z}{w} \)
  - \( \overline{z} = x - iy \) \hspace{1cm} \text{conjugate}
  - \( \text{Re}(z) = x \) \hspace{1cm} \text{real part}
  - \( \text{Im}(z) = y \) \hspace{1cm} \text{imaginary part}
  - \( |z| = \sqrt{x^2 + y^2} \) \hspace{1cm} \text{modulus}
  - \( |z|^2 = \overline{z}z = \sqrt{z\overline{z}} \)

This is basic arithmetic we expect you already know. Now for Chapter 2...
There are three primary representations of complex numbers.

- **rectangular**: \( x + iy \)
- **polar**: \( r (\cos(\theta) + i \sin(\theta)) \)
- **exponential**: \( re^{i\theta} \)

The polar form of a complex number is closely linked to polar coordinate:

\[ x + iy = r \cos(\theta) + i r \sin(\theta) \]

Note that \( r = \sqrt{x^2 + y^2} = |z| \)
Convention: We will usually allow $\theta$-values only in $[0, 2\pi)$.

The exponential form ties several topics in Calculus II together.

What does $e^{3i}$ mean?

Let's look at the Taylor series for $e^z$.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

(valid for all real $z$)

If $e^{i\theta}$ is to make any sense at all, we should have

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$i^0 = 1$
$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$
$i^6 = -1$
$i^7 = -i$
The powers of $i$ repeat in a cycle of 4: $1, i, -1, -i$

Let's use this to rewrite the power series for $e^{i\theta}$!

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \frac{i^7\theta^7}{7!} + \ldots$$

$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \ldots$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right)$$
Do you recognize these power series?

\[ e^{i\theta} = \cos(\theta) + i\sin(\theta) \]

(Euler's identity)

So the polar and exponential forms of a complex number are linked

\[ z = r\cos(\theta) + i\, r\sin(\theta) = re^{i\theta} \]

This makes multiplication/division easy

\[ e^{i\theta} e^{i\varphi} = e^{i(\theta + \varphi)} \]

\[ (e^{i\theta})^n = e^{in\theta} \]

(All usual rules of exponents still hold.)

---

Ex. 1

Find all forms of each complex number.

Solution:

(a) (Rect.) \[ z = -2\sqrt{2} + 2\sqrt{2}i \]
(a) \( r = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{8 + 8} = \sqrt{16} = 4 \)
\[ \tan(\theta) = \frac{2\sqrt{2}}{-2\sqrt{2}}, \quad \theta = \frac{3\pi}{4} \]

Polar: \( z = 4 \cos \left(\frac{3\pi}{4}\right) + i 4 \sin \left(\frac{3\pi}{4}\right) \)

Exponential: \( z = 4 e^{i 3\pi/4} (r e^{i \theta}) \)

(b) (Rect.) \( z = -1 \quad (-1 + 0i) \)

Polar: \( z = 1 \cos(\pi) + i 1 \sin(\pi) \)

Exponential: \( z = 1 e^{i \pi} \)

(c) (Rect.) \( z = i \quad (0 + 1i) \)

Polar: \( z = \cos(\frac{\pi}{2}) + i \sin(\frac{\pi}{2}) \)

Exponential: \( z = e^{i \pi/2} \)
DeMoivre's Formula

\[(e^{i\theta})^n = e^{in\theta}\]

\[(\cos \theta + i\sin \theta)^n = \cos(n\theta) + i\sin(n\theta)\]

This helps with trigonometric identities.

Ex. 2

Use DeMoivre's Formula to derive a triple angle formula for \(\sin(3\theta)\) and \(\cos(3\theta)\).

**Solution**:

We have that

\[(\cos \theta + i\sin \theta)^3 = \cos(3\theta) + i\sin(3\theta) \quad (1)\]

Expand the left side using algebra.

\[(\cos \theta + i\sin \theta)^3 = \cos^3 \theta + 3\cos^2 \theta \cdot i\sin \theta + \]

\[+ 3\cos \theta \cdot i^2 \sin^2 \theta + i^3 \sin^3 \theta\]

\[= [\cos^3 \theta - 3\cos \theta \sin^2 \theta] + i [3\cos^2 \theta \sin\theta - \sin^3 \theta]\]
Now back to Equation (i).

\[
[\cos^3 \theta - 3\cos \theta \sin^2 \theta] + i [3\cos^2 \theta \sin \theta - \sin^3 \theta]
= \cos(3\theta) = \sin(3\theta)
\]

So we obtain the formulas:

\[
\cos(3\theta) = \cos(\theta)^3 - 3\cos(\theta)\sin(\theta)^2 \\
\sin(3\theta) = 3\cos(\theta)^2\sin(\theta) - \sin(\theta)^3
\]

Using \(\cos(\theta)^2 + \sin(\theta)^2 = 1\), these formulas are more commonly written as:

\[
\cos(3\theta) = 4\cos(\theta)^3 - 3\cos(\theta) \\
\sin(3\theta) = 3\sin(\theta) - 4\sin(\theta)^3
\]

**Finding Complex Roots**

Our primary goal now is to solve equations of the form

\[z^n = w\]

where \(n > 0\) is a given integer and \(w\) is a given complex number. In general,
there are $n$ distinct solutions for $z$. We say $z$ is a $n$th root of $w$. We will see that one complication is that the function $f(z) = e^z$ is not one-to-one for complex numbers.

There is no such thing as “ln” for complex numbers!

**Theorem #1:**

Suppose $e^z = 1$. Then there is an integer $N$ such that $z = 2\pi i N$.

**Proof:**

Let $z = x + i y$ with $x$ and $y$ real.

$$1 = e^z = e^{x + iy} = e^x e^{iy}$$

Now take the modulus of both sides

$$1 = |1| = |e^x e^{iy}| = |e^x| \cdot |e^{iy}| = e^x \cdot |e^{iy}|$$

Note that $e^{iy} = \cos(y) + i \sin(y)$. So...
\[ |e^{iy}| = \sqrt{\cos(y)^2 + \sin(y)^2} = 1 \]

So now we have:

\[ 1 = e^x \]

Since \( x \) is real, we have \( x = 0 \), and we are left with

\[ 1 = e^{iy} = \cos(y) + i\sin(y) \]

This is equivalent to the system:

\[ \cos(y) = 1 \]
\[ \sin(y) = 0 \]

The solutions to this system are \( y = 2\pi N \) where \( N \) is any integer. Hence we conclude \( z = iy = 2\pi i N \).

This theorem explicitly shows that \( e^z \) is not one-to-one. How do we solve \( e^z = e^w \)?

**Theorem 2**

Suppose \( e^z = e^w \). Then there is an integer \( N \) such that \( z = w + 2\pi i N \).
Proof:
If $e^z = e^w$, then $e^{z - w} = 1$ by usual rules of exponents. From Theorem #1, we have

$$z - w = 2\pi i N$$

for some integer $N$. So $z = w + 2\pi i N$.  

Theorem #2 is crucial in finding roots of complex numbers.

---

**Ex. 2**

Find all solutions to $z^3 = 8i$.

**Solution:**
First write both sides of the equation in exponential form. Suppose

$$z = re^{i\theta}$$

where $r$ and $\theta$ are unknown. Now write $8i$ in exponential form.

$$8i = Re^{i\varphi}$$
So we have

\[8i = 8e^{i\pi/2}\]

The equation \(z^3 = 8i\) is now written as:

\[z^3 = 8i\]

\[(re^{i\theta})^3 = 8e^{i\pi/2}\]

\[r^3 e^{i3\theta} = 8e^{i\pi/2}\]

\[r = 2\]

\[i \cdot 3\theta = i \frac{\pi}{2} + 2\pi i N\]

(\textbf{Theorem \#2})

Note the use of Theorem \#2! So now

\[\theta = \frac{\pi}{6} + \frac{2\pi}{3} \cdot N\]

So all solutions have the form:
\[ z = r e^{i \theta} = 2 e^{i \left( \frac{\pi}{6} + \frac{2\pi}{3} N \right)} \]
\[ = 2 \cos \left( \frac{\pi}{6} + \frac{2\pi}{3} N \right) + i \cdot 2 \sin \left( \frac{\pi}{6} + \frac{2\pi}{3} N \right) \]

where \( N \) is an integer. This looks like there are infinitely many solutions. But the periodicity of cosine and sine implies there are only 3 distinct solutions.

\( N = 0 \): \[ z_1 = 2 e^{i \pi/6} = 2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) = \sqrt{3} + i \]

\( N = 1 \): \[ z_2 = 2 e^{i 5\pi/6} = 2 \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) = \sqrt{3} - i \]

\( N = 2 \): \[ z_3 = 2 e^{i 9\pi/6} = 2 \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) = 2 (0 - i) = -2i \]

**Ex. 3**

Find all complex solutions to
\[ z^3 = -2 - 2\sqrt{3} \ i \]
Solution:

Let \( z = re^{i\theta} \) (\( r, \theta \) unknown). Convert \( w = -2 - 2\sqrt{3}i \) to exponential form.

\[
|w| = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = 4
\]

\[
\tan(\phi) = \frac{-2\sqrt{3}}{-2} = \sqrt{3}
\]

\[
\phi = \frac{4\pi}{3}
\]

Now substitute into our equation.

\[
z^3 = -2 - 2\sqrt{3}i
\]

\[
(\sqrt{r}e^{i\phi})^3 = 4e^{i\frac{4\pi}{3}}
\]

\[
r^3 e^{i3\phi} = 4e^{i\frac{4\pi}{3}}
\]

\[
r^3 = 4
\]

\[
e^{i3\phi} = e^{i\frac{4\pi}{3}}
\]

\[
i3\phi = i\frac{4\pi}{3} + 2\pi i N
\]

\[
\theta = \frac{4\pi}{9} + \frac{2\pi}{3} N
\]
So our three unique solutions are

\[ N=0 : \quad z_1 = 4^{1/3} e^{i\pi/9} = 4^{1/3} \left( \cos \frac{4\pi}{9} + i \sin \frac{4\pi}{9} \right) \]

\[ N=1 : \quad z_2 = 4^{1/3} e^{10\pi i/9} = 4^{1/3} \left( \cos \frac{10\pi}{9} + i \sin \frac{10\pi}{9} \right) \]

\[ N=2 : \quad z_3 = 4^{1/3} e^{16\pi i/9} = 4^{1/3} \left( \cos \frac{16\pi}{9} + i \sin \frac{16\pi}{9} \right) \]

---

**Geometric Interpretation of Multiplication**

Consider the following:

\[ z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2} \]

\[ z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \]

So lengths multiply and angles add.
If \( \omega^n = 1 \), we say \( \omega \) is an \( n \)th root of unity. What do these solutions look like? (Let \( \omega = re^{i\theta} \).)

\[
\omega^n = 1 \\
r^n e^{i\theta} = 1 \cdot e^{i0}
\]

\[
r^n = 1 \\
e^{i\theta} = e^{i0}
\]

\[
r = 1 \\
\theta = 0 + 2\pi i N
\]

where \( N \) is an integer \( \rightarrow \theta = 2\pi \frac{N}{n} \)

Hence the \( n \) \( n \)th roots of unity are:

\( N=0 : \omega_1 = 1 \)

\( N=1 : \omega_2 = \cos \left(2\pi \cdot \frac{1}{n}\right) + i \sin \left(2\pi \cdot \frac{1}{n}\right) \)

\( N=2 : \omega_3 = \cos \left(2\pi \cdot \frac{2}{n}\right) + i \sin \left(2\pi \cdot \frac{2}{n}\right) \)

\( \vdots \)

\( N=n-1 : \omega_n = \cos \left(2\pi \cdot \frac{n-1}{n}\right) + i \sin \left(2\pi \cdot \frac{n-1}{n}\right) \)
Note that $|\omega_k| = 1$ for all $k$ and there are $n$ solutions, each separated by the angle $2\pi/n$. So what if we plot these points in the complex plane?

In general, the $n$th roots of unity are the vertices of a regular $n$-gon inscribed in the unit circle, with one vertex at $\omega = 1$.

Finally, let $\omega_1 = e^{i \cdot 2\pi/n}$. Then by rules...
of exponents:

$$\omega_k = e^{i\frac{2\pi k}{n}} = (e^{i\frac{2\pi}{n}})^k = \omega_1^k$$

If you know one root of unity (not 1), you get the others by higher powers:

$$1, \omega_1, \omega_1^2, \omega_1^3, \ldots, \omega_1^{n-1}$$

The $n$ $n$th roots of unity

This offers an alternative way of finding complex roots. Suppose we want to solve:

$$z^n = w$$

Let $z_0$ be any solution to this equation. That is, $z_0^n = w$. Now let $\omega$ be an $n$th root of unity.

$$\omega^n = 1$$

Then $z = \omega z_0$ solves our equation!

$$(\omega z_0)^n = \omega^n z_0^n = 1 \cdot w = w$$
So if you can find any solution to $z^n = w$ and any $n$th root of unity $\omega$, the $n$ solutions to $z^n = w$ are:

Solution 1: $z_1 = z_0$
Solution 2: $z_2 = \omega z_0$
Solution 3: $z_3 = \omega^2 z_0$
Solution 4: $z_4 = \omega^3 z_0$

$\vdots$

Solution #n: $z_n = \omega^{n-1} z_0$

**Ex:**

Given that $(1+i)^7 = 8-8i$, find all solutions to $z^7 = 8-8i$.

Solution:

Let $\omega$ be any 7th root of unity except 1. One choice is $\omega = e^{i\frac{2\pi}{7}}$.

(Check: $\omega^7 = (e^{i\frac{2\pi}{7}})^7 = e^{i2\pi} = \cos(2\pi) + i\sin(2\pi) = 1$)
One solution to $z^7 = 8 - 8i$ is
\[ Z_0 = 1 + i \]
Thus the 7 solutions to $z^7 = 8 - 8i$ are:
\[
\begin{align*}
Z_1 &= Z_0 = 1 + i \\
Z_2 &= \omega Z_0 = (\cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7})(1 + i) \\
Z_3 &= \omega^2 Z_0 = (\cos \frac{4\pi}{7} + i \sin \frac{4\pi}{7})(1 + i) \\
Z_4 &= \omega^3 Z_0 = \ldots \\
Z_5 &= \omega^4 Z_0 = \ldots \\
Z_6 &= \omega^5 Z_0 = \ldots \\
Z_7 &= \omega^6 Z_0 = \ldots 
\end{align*}
\]
Hyperbolic Functions Reference Sheet

Basic Definitions and Graphs

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\text{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$
Inverse Hyperbolic Functions and Graphs

\[ \text{csch}(x) = \frac{1}{\sinh(x)} \]
\[ = \frac{2}{e^x - e^{-x}} \]

\[ \text{coth}(x) = \frac{\cosh(x)}{\sinh(x)} \]
\[ = \frac{e^x + e^{-x}}{e^x - e^{-x}} \]

\[ y = \cosh^{-1}(x) \]
\[ \allowdisplaybreaks \]
\[ \cosh(y) = x \quad \text{and} \quad 0 \leq y < \infty \]
\[ \text{domain:} \quad 1 \leq x < \infty \]
\[ \cosh^{-1}(x) = \ln \left( x + \sqrt{x^2 - 1} \right) \]

\[ y = \sinh^{-1}(x) \]
\[ \allowdisplaybreaks \]
\[ \sinh(y) = x \]
\[ \text{domain:} \quad -\infty < x < \infty \]
\[ \sinh^{-1}(x) = \ln \left( x + \sqrt{x^2 + 1} \right) \]
\[ y = \tanh^{-1}(x) \]
\[ \tanh(y) = x \]
\[ \text{domain: } -1 < x < 1 \]
\[ \tanh^{-1}(x) = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) \]

\[ y = \sech^{-1}(x) \]
\[ \sech(y) = x \text{ and } 0 \leq y < \infty \]
\[ \text{domain: } 0 < x \leq 1 \]
\[ \sech^{-1}(x) = \ln \left( \frac{1 + \sqrt{1-x^2}}{x} \right) \]

\[ y = \csch^{-1}(x) \]
\[ \csch(y) = x \]
\[ \text{domain: } |x| > 0 \]
\[ \csch^{-1}(x) = \ln \left( \frac{1 + \sqrt{1+x^2}}{x} \right) \]

\[ y = \coth^{-1}(x) \]
\[ \coth(y) = x \]
\[ \text{domain: } |x| > 1 \]
\[ \coth^{-1}(x) = \frac{1}{2} \ln \left( \frac{x+1}{x-1} \right) \]
Derivatives

\[
\frac{d}{dx} \cosh(x) = \sinh(x)
\]

\[
\frac{d}{dx} \sinh(x) = \cosh(x)
\]

\[
\frac{d}{dx} \tanh(x) = \text{sech}(x)^2
\]

\[
\frac{d}{dx} \text{sech}(x) = -\text{sech}(x) \tanh(x)
\]

\[
\frac{d}{dx} \text{csch}(x) = -\text{csch}(x) \coth(x)
\]

\[
\frac{d}{dx} \coth(x) = -\text{csch}(x)^2
\]

\[
\frac{d}{dx} \text{cosh}^{-1}(x) = \frac{1}{\sqrt{x^2-1}}
\]

\[
\frac{d}{dx} \sinh^{-1}(x) = \frac{1}{\sqrt{x^2+1}}
\]

\[
\frac{d}{dx} \tanh^{-1}(x) = \frac{1}{1-x^2}
\]

\[
\frac{d}{dx} \text{sech}^{-1}(x) = -\frac{1}{x \sqrt{1-x^2}}
\]

\[
\frac{d}{dx} \text{csch}^{-1}(x) = -\frac{1}{|x| \sqrt{x^2+1}}
\]

\[
\frac{d}{dx} \coth^{-1}(x) = \frac{1}{1-x^2}
\]

Identities and Triangles

\[
\cosh(x)^2 - \sinh(x)^2 = 1
\]

\[
1 - \tanh(x)^2 = \text{sech}(x)^2
\]

\[
\coth(x)^2 - 1 = \text{csch}(x)^2
\]

\[\cosh(\theta) = \frac{\text{ADJ.}}{\text{HYP.}} \quad \text{sech}(\theta) = \frac{\text{HYP.}}{\text{ADJ.}}\]

\[\sinh(\theta) = \frac{\text{OPP.}}{\text{HYP.}} \quad \text{csch}(\theta) = \frac{\text{HYP.}}{\text{OPP.}}\]

\[\tanh(\theta) = \frac{\text{OPP.}}{\text{ADJ.}} \quad \coth(\theta) = \frac{\text{ADJ.}}{\text{OPP.}}\]

\[(\text{ADJACENT})^2 - (\text{OPPOSITE})^2 = (\text{HYPOTENUSE})^2\]
7.1: Integration by Parts

\[ \int u \, dv = uv - \int v \, du \]

Priority for choosing u:

- L: logarithms
- I: inverse trigonometric, inverse hyperbolic
- A: algebraic
- T: trigonometric, hyperbolic
- E: exponential

7.2: Trigonometric Integrals

\[ \int \sin(x)^m \cos(x)^n \, dx \]

(A) \( m \) odd (n anything)

- Split off factor of \( \sin(x) \)
- Rewrite remaining powers of \( \sin(x) \) in terms of \( \cos(x) \) using identity \( \sin(x)^2 = 1 - \cos(x)^2 \)
• use the substitution \( u = \cos(x) \)

(B) \( n \) odd (\( m \) anything)

• split off factor of \( \cos(x) \)

• rewrite remaining powers of \( \cos(x) \) in terms of \( \sin(x) \) using identity \( \cos(x)^2 = 1 - \sin(x)^2 \)

• use the substitution \( u = \sin(x) \)

(c) \( m \) and \( n \) both even

• rewrite entire integrand in terms of \( \sin(x) \) only or \( \cos(x) \) only using identity \( \cos(x)^2 + \sin(x)^2 = 1 \).

• If rewritten in terms of \( \sin(x) \)...

• use integration by parts with 
  \( dv = \sin(x) \, dx \)

• in resulting integral, rewrite \( \cos(x)^2 \) as \( 1 - \sin(x)^2 \)

• algebraically solve for original integral.

• If rewritten in terms of \( \cos(x) \)...

• use the substitution \( u = \cos(x) \)
• use integration by parts with 
  \( dv = \cos(x) \, dx \)
• in resulting integral, rewrite \( \sin(x)^2 \) as \( 1 - \cos(x)^2 \)
• algebraically solve for original integral.

\[
\int \tan(x)^m \sec(x)^n \, dx
\]

(A) Special cases (memorize)
• \( \int \tan(x) \, dx = \ln | \sec(x) | + C \)
• \( \int \sec(x) \, dx = \ln | \sec(x) + \tan(x) | + C \)

(B) \( m \) odd \ (n \) anything
• split off factor of \( \sec(x) \tan(x) \)
• rewrite remaining powers of \( \tan(x) \)
in terms of \( \sec(x) \) using identity
  \( \tan(x)^2 = \sec(x)^2 - 1 \)
• use the substitution $u = \sec(x)$

(C) $m$ even and $n \neq 2$ (m anything)

• split off factor of $\sec(x)^2$
• rewrite remaining powers of $\sec(x)$ in terms of $\tan(x)$ using identity
  $\sec(x)^2 = \tan(x)^2 + 1$
• use the substitution $u = \tan(x)$

(D) $m$ even and $n$ odd

• rewrite entire integrand in terms of $\sec(x)$ only using identity
  $\tan(x)^2 = \sec(x)^2 - 1$
• use integration by parts with
  $dv = \sec(x)^2 \, dx$
• in resulting integral, rewrite $\tan(x)^2$ as $\sec(x)^2 - 1$
• algebraically solve for original integral:

$$\int \cot(x)^m \csc(x)^n \, dx$$
(A) special cases (memorize)
  • \( \int \cot(x) \, dx = -\ln |\csc(x)| + C \)
  • \( \int \csc(x) \, dx = -\ln |\csc(x) + \cot(x)| + C \)

(B) \( m \) odd (\( n \) anything)
  • split off factor of \( \csc(x) \cot(x) \)
  • rewrite remaining powers of \( \cot(x) \) in terms of \( \csc(x) \) using identity \( \cot(x)^2 = \csc(x)^2 - 1 \)
  • use the substitution \( u = \csc(x) \)

(C) \( n \) even and \( n \neq 2 \) (\( m \) anything)
  • split off factor of \( \csc(x)^2 \)
  • rewrite remaining powers of \( \csc(x) \) in terms of \( \cot(x) \) using identity \( \csc(x)^2 = \cot(x)^2 + 1 \)
  • use the substitution \( u = \cot(x) \)

(D) \( m \) even and \( n \) odd
* rewrite entire integrand in terms of \( \csc(x) \) only using identity \( \cot(x)^2 = \csc(x)^2 - 1 \)
* use integration by parts with \( du = \csc(x)^2 \, dx \)
* in resulting integral, rewrite \( \cot(x)^2 \) as \( \csc(x)^2 - 1 \)
* algebraically solve for original integral.

**7.3: Trigonometric Substitution**

* Use trigonometric substitution for integrands with quadratic expressions under some integer power of a square root.
* Complete the square as necessary to obtain one of the following forms:
  \[
  \begin{cases}
  \sqrt{a^2 - x^2} \\
  \sqrt{a^2 + x^2} \\
  \sqrt{x^2 - a^2}
  \end{cases}
  \]
  always assume \( a > 0 \).
Use the table below as necessary:

<table>
<thead>
<tr>
<th>Expression</th>
<th>Substitution</th>
<th>Identities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{a^2-x^2}$</td>
<td>$x = \sin(\theta)$</td>
<td>$dx = \cos(\theta),d\theta$</td>
</tr>
<tr>
<td></td>
<td>$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$</td>
<td>$\sqrt{a^2-x^2} = \cos(\theta)$</td>
</tr>
</tbody>
</table>

| $\sqrt{a^2+x^2}$ | $x = \tan(\theta)$ | $dx = \sec^2(\theta)\,d\theta$ |
|                  | $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ | $\sqrt{a^2+x^2} = \sec(\theta)$ |

| $\sqrt{x^2-a^2}$ | $x = \sec(\theta)$ | $dx = \sec(\theta)\tan(\theta)\,d\theta$ |
|                  | $0 \leq \theta < \frac{\pi}{2}$ OR $\pi \leq \theta < \frac{3\pi}{2}$ | $\sqrt{x^2-a^2} = \tan(\theta)$ |

7.4: Integrals with Hyperbolic Functions

Note: Be sure to go over the “Hyperbolic Functions Review Sheet”.

(A) Substitution

(B) Integration by Parts

When choosing $u$, treat hyperbolic and inverse hyperbolic functions
as you would treat trigonometric and inverse trigonometric functions

(C) Hyperbolic Integrals

Treat powers of hyperbolic functions as you would treat trigonometric functions. The strategies are identical

(D) Hyperbolic Substitution

Similar to trigonometric substitution

<table>
<thead>
<tr>
<th>Expression</th>
<th>Trig. Substitution</th>
<th>Hyp. Substitution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{a^2-x^2} )</td>
<td>( x = \sin(\theta) )</td>
<td>( x = \tanh^{-1}(u) )</td>
</tr>
<tr>
<td>( \sqrt{a^2+x^2} )</td>
<td>( x = \tan(\theta) )</td>
<td>( x = \sinh^{-1}(u) )</td>
</tr>
<tr>
<td>( \sqrt{x^2-a^2} )</td>
<td>( x = \sec(\theta) )</td>
<td>( x = \cosh^{-1}(u) )</td>
</tr>
</tbody>
</table>

7.5: Method of Partial Fractions

- Partial fractions can be used for integrating any rational function

\[
f(x) = \frac{P(x)}{Q(x)} \quad \text{P and Q are polynomials with no common factors}\]
• If \( \deg(P) \geq \deg(Q) \), perform long division first to write

\[
f(x) = R(x) + \frac{r(x)}{Q(x)}
\]

polynomial of degree remainder term:
\( \deg(P) - \deg(Q) \quad \deg(r) < \deg(Q) \)

• Find the partial fraction decomposition (PFD) of the remainder term
  • Factor \( Q(x) \) into irreducible factors
    • \( ax + b \quad \text{OR} \)
    • \( ax^2 + bx + c \quad \text{(with} \ b^2 - 4ac < 0) \)
  • Write \( \frac{r(x)}{Q(x)} \) as a sum of simple partial fractions. If a factor of \( Q(x) \) is repeated \( m \) times, then the PFD has \( m \) terms for that repeated factor.
The numerator of each term should be one degree less than the corresponding factor of \( Q \).

**Examples:**

\[
\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}
\]

\[
\frac{1}{(x-a)(x-b)^2} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-b)^2}
\]

\[
\frac{1}{(x-a)(x^2+b^2)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+b^2}
\]

• Find the values of the unknown coefficients using algebra

• Find the desired integral using the PFD. For quadratic factors, split into two: one part uses substitution and the other uses trig substitution \((x = a \tan(\theta))\)
Some common integrals that are useful to memorize:

\[ \int \frac{x}{x^2 + a^2} \, dx = \frac{1}{2} \ln (x^2 + a^2) + C \]

\[ \int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C \]
Special Integrals:

(A) \[ \int \sec(x) \, dx = \ln |\sec(x) + \tan(x)| + C \]
\[ = \sinh^{-1}(\tan(x)) + C \]
\[ = \tanh^{-1}(\sin(x)) + C \]

(B) \[ \int \csc(x) \, dx = -\ln |\csc(x) + \cot(x)| + C \]
\[ = -\sinh^{-1}(\cot(x)) + C \]
\[ = -\tanh^{-1}(\cos(x)) + C \]

(C) \[ \int \sech(x) \, dx = \ldots \ldots \]
\[ = \sin^{-1}(\tanh(x)) + C \]
\[ = \tan^{-1}(\sinh(x)) + C \]

(D) \[ \int \csch(x) \, dx = -\ln |\csch(x) + \coth(x)| + C \]
\[ = -\cosh^{-1}(\coth(x)) + C \]
\[ = -\tanh^{-1}(\cosh(x)) + C \]
Finding Coefficients for Partial Fractions

In the 7.5 notes, we found that
\[ \frac{1}{x^2-7x+10} = \frac{1/3}{x-5} - \frac{1/3}{x-2} \]

Of course, we can check that this is true, but why did our method work? From the equation
\[ \frac{1}{x^2-7x+10} = \frac{A}{x-5} + \frac{B}{x-2} \]  \hspace{1cm} (1)

we have
\[ 1 = A(x-2) + B(x-5) \]  \hspace{1cm} (2)

Then we substituted \( x = 2 \) and \( x = 5 \). But (2) is not valid for \( x = 2 \) or \( x = 5 \) since (1) is not valid. So why can we do that anyway and get the correct answer?
The answer is limits and continuity!

Multiplying (1) by \( x^2 - 7x + 10 \) gives

\[
1 = \left( \frac{A}{x-5} + \frac{B}{x-2} \right) (x^2 - 7x + 10)
\]

\[
= f(x)
\]

Obviously, \( f(2) \) and \( f(5) \) are not defined, but these are removable discontinuities. What about the limits of \( f(x) \)?

\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} \left[ \left( \frac{A}{x-5} + \frac{B}{x-2} \right) (x^2 - 7x + 10) \right]
\]

\[
= \lim_{x \to 2} \left[ A(x-2) + B(x-5) \right]
\]

\[
= 0 + B(-3) = -3B
\]

But since \( f(x) = 1 \) for all \( x \neq 2, 5 \) we have \( -3B = 1 \), or \( B = -\frac{1}{3} \).
So when we substitute $x=2$ or $x=5$ into (2) to find $A$ and $B$, we are really implicitly computing limits!
Some Peculiarities with Complex Numbers

Consider the integral
\[ \int \frac{1}{x^2 + 1} \, dx \]

We know this is \( \tan^{-1}(x) + C \). But partial fractions gives a different answer.

The polynomial \( p(x) = x^2 + 1 \) is not reducible over the real numbers, but it is reducible over the complex numbers. (The fundamental theorem of algebra tells us all polynomials of degree 2 or higher are reducible over the complex numbers.)

\[ x^2 + 1 = (x+i)(x-i) \text{ where } i^2 = -1 \]

So we have the PFD:
\[
\frac{1}{x^2+1} = \frac{A}{x+i} + \frac{B}{x-i}
\]

Find the coefficients by the usual method.

\[1 = A(x-i) + B(x+i)\]

\[x = i : 1 = A(0) + B(2i) \Rightarrow B = \frac{1}{2i}\]

\[x = -i : 1 = A(-2i) + 0 \Rightarrow A = \frac{1}{2i}\]

So our integral is

\[
\int \frac{1}{x^2+1} \, dx = \int \frac{1}{2i} \left( \frac{-1}{x+i} + \frac{1}{x-i} \right) \, dx
\]

\[= \frac{1}{2i} \left( -\ln(x+i) + \ln(x-i) \right) + C' \]

\[= \frac{1}{2i} \ln \left( \frac{x-i}{x+i} \right) + C' \]

This doesn't look like \(\tan^{-1}(x)\)?! We have found evidently that there is some constant \(C\) such that

\[\tan^{-1}(x) = \frac{1}{2i} \ln \left( \frac{x-i}{x+i} \right) + C\]
We can find the value of $C$ by considering $x \to -\infty$:

$$\lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$$

$$\lim_{x \to -\infty} \frac{1}{2i} \ln \left( \frac{x-i}{x+i} \right) = \frac{1}{2i} \ln (1) = 0$$

So we find that $C = -\frac{\pi}{2}$, hence

$$\tan^{-1}(x) = \frac{1}{2i} \ln \left( \frac{x-i}{x+i} \right) - \frac{\pi}{2}$$

Very peculiar! What if $x = 0$?

$$0 = \frac{1}{2i} \ln (-1) - \frac{\pi}{2}$$

Hmm... now rearrange the equation:

$$0 = \ln (-1) - i\pi$$

$$i\pi = \ln (-1)$$

$$e^{i\pi} = -1$$

$$e^{i\pi} + 1 = 0$$

Whatever all that means!
Applications of the Integral

6.1: Area Between Curves

\[ A = \int_a^b (f(x) - g(x)) \, dx \quad A = \int_c^d (f(y) - g(y)) \, dy \]

6.2: Volume, Density, Average Value

- **Average Value of** \( f(x) \) **on** \([a, b] \):
  \[ M = \frac{1}{b-a} \int_a^b f(x) \, dx \]

- **Total mass (charge, population, etc.)** given linear density \( \lambda(x) \):
  \[ m = \int_a^b \lambda(x) \, dx \]

- **Total mass (charge, population, etc.)** given radial density \( \sigma(r) \):
\[ m = \int_{r_i}^{r_f} 2\pi r \sigma(r) \, dr \]

- Volume of solid for which cross-sections perpendicular to x-axis have area \( A(x) \) at coordinate \( x \).

\[ V = \int_a^b A(x) \, dx \]

### 6.3: Volumes of Revolution

- Method of Washers:

\[ V = \int_a^b \pi \left( (\text{Outer})^2 - (\text{Inner})^2 \right) \, dh \]

- Volume by slicing: cross section is perpendicular to axis of rotation
  - For vertical slice, \( dh = dx \)
  - For horizontal slice, \( dh = dy \)

### 6.4: Method of Cylindrical Shells

\[ V = \int_a^b 2\pi Rh \, dr \]
volume obtained by rotating rectangular strips to form nested shells.

strip is parallel to axis of rotation

$F$: average radius of shell (distance from rotation axis to strip)

$h$: height of shell (length of strip)

For vertical strip, $dr = dx$

For horizontal strip, $dr = dy$

8.1: Arc Length and Surface Area

Length of curve $y = f(x)$ over $[a, b]$.

$$s = \int_a^b \sqrt{1 + f'(x)^2} \, dx$$

Length of curve $x = f(y)$ over $[c, d]$.

$$s = \int_c^d \sqrt{1 + f'(y)^2} \, dy$$

Arc length differential

$$ds = \sqrt{(dx)^2 + (dy)^2}$$
\[
\sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

\[
\sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy
\]

- **Area of surface obtained by rotating the curve** \( y = f(x) \) \( (a \leq x \leq b) \) **about the** \( x \)-**axis**: 
  
  \[
  A = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} \, dx
  \]

- **General surface area formula:**
  
  \[
  A = \int_a^b 2\pi \overline{r} \, ds
  \]

- **Area obtained by rotating conical bands about axis of rotation.**
  
  - \( \overline{r} \): average radius of band (distance from rotation axis to strip)
  - \( ds \): arc length differential
Consider the curve $y = f(x)$ for $a \leq x \leq b$.

Partition $[a, b]$ into $N$ subintervals with $a = x_0 < x_1 < x_2 < \ldots < x_N = b$. Approximate the curve by a polygonal curve whose vertices are the points $P_k = (x_k, f(x_k))$. Let $L_k = |P_{k-1}P_k|$. Then we have

$$L_k = \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2} = \Delta x_k \Delta y_k$$

Now suppose $f'(x)$ is continuous. By the
mean value theorem, for each interval \([x_{k-1}, x_k]\) there is a number \(c_k\) with

\[
\frac{\Delta y_k}{\Delta x_k} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(c_k)
\]

\[\Rightarrow \Delta y_k = f'(c_k) \Delta x_k\]

Hence we have

\[L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{1 + f'(c_k)^2} \Delta x_k\]

The total length of the polygonal curve is given by

\[L_{\text{total}} = \sum_{k=1}^{N} \sqrt{1 + f'(c_k)^2} \Delta x_k\]

This is a Riemann sum for the integral of \(\sqrt{1 + f'(x)^2}\) on \([a, b]\), with sampling points \(c_1, c_2, \ldots, c_N\). The exact length of the curve is thus

\[s = \lim_{N \to \infty} L_{\text{total}} = \int_a^b \sqrt{1 + f'(x)^2} \, dx\]
Derivation of Area of Conical Band

First consider a right circular cone with base radius $R$ and slant height $L$.

Cut cone along dashed line (along its side) and roll out.

The outer perimeter is $2\pi R$ (since it was the circumference of the cone's base). If the disk were complete (not missing wedge), the circumference would be $2\pi L$. 
The ratio of the incomplete circumference to the complete circumference is the same as for the area:

\[
\frac{A_{\text{incomplete}}}{A_{\text{complete}}} = \frac{C_{\text{incomplete}}}{C_{\text{complete}}} = \frac{2\pi R}{2\pi L} = \frac{R}{L}
\]

But \(A_{\text{incomplete}}\) is just the area of the cone and \(A_{\text{complete}}\) is \(\pi L^2\). So now we have

\[
A_{\text{cone}} = (\pi L^2) \cdot \frac{R}{L} = \pi R L
\]

Now we use this formula to find the area of a conical band, with base radii \(R_1\) and \(R_2\) and slant height \(L\).
We consider the band as the difference of two cones:

Cone #1: radius = \( R_1 \),
slant height = \( L' \)

Cone #2: radius = \( R_2 \),
slant height = \( L + L' \)

(Of course, our final answer should contain only \( R_1 \), \( R_2 \), and \( L \), the dimensions of the band.) By similar triangles,

\[
\frac{L'}{R_1} = \frac{L + L'}{R_2}
\]

With some algebra, we find

\[
L' = \frac{R_1 L}{R_2 - R_1}
\]

Using our formula for the area of a cone and our formula for \( L' \), we have

\[
A_{\text{band}} = A_{\text{cone #2}} - A_{\text{cone #1}}
\]

\[
A_{\text{band}} = \pi R_2 (L + L') - \pi R_1 L'
\]

\[
A_{\text{band}} = \pi R_2 L + \pi (R_2 - R_1) L'
\]
This is our desired formula for the area of a conical band (truncated cone).

\[ A_{\text{band}} = \pi R_2 L + \pi R_1 L \]

\[ A_{\text{band}} = \pi (R_1 + R_2) L \]
Derivation of Surface Area Formula

We begin with the same setup as in the derivation for the arc length formula.

The area of revolution is approximated by the surface obtained by rotating the polygonal curve about the x-axis. So consider the conical band obtained by rotating the segment $P_{k-1}P_k$.

$X_{k-1}$: left x-coordinate
$X_k$: right x-coordinate

$y_{k-1} = f(x_{k-1})$
$y_k = f(x_k)$

$L_k = |P_{k-1}P_k|$

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$
As with the arc length derivation, we suppose \( f'(x) \) is continuous. So by the mean value theorem there exists \( c_k \) in \((x_{k-1}, x_k)\) such that

\[
L_k = \sqrt{1 + f'(c_k)^2} \Delta x_k
\]

Using the formula for the area of a conical band, we find the area of the \( k \)th band is

\[
A_k = \pi \left( f(x_{k-1}) + f(x_k) \right) \sqrt{1 + f'(c_k)^2} \Delta x_k
\]

Hence the total area of the polygonal surface of revolution is

\[
A_{\text{total}} = \sum_{k=1}^{N} \pi \left( f(x_{k-1}) + f(x_k) \right) \sqrt{1 + f'(c_k)^2} \Delta x_k
\]

This is not quite a Riemann sum since the terms involve three sample points in the interval \([x_{k-1}, x_k]\) (namely, the samples at \( x_{k-1}, x_k, \) and \( c_k \)). A more advanced argument, which requires continuity of \( f'(x) \) shows that we may replace \( x_{k-1} \) and \( x_k \) with \( c_k \) without
affecting the limit as \( N \to \infty \).

(This argument is beyond the scope of 152.)

So the exact surface area is:

\[
A = \lim_{N \to \infty} \left( \sum_{k=1}^{N} \pi \left( f(c_k) + f(c_k) \right) \sqrt{1 + f'(c_k)^2} \, \Delta x_k \right)
\]

\[
= \lim_{N \to \infty} \left( \sum_{k=1}^{N} 2\pi f(c_k) \sqrt{1 + f'(c_k)^2} \, \Delta x_k \right)
\]

Now this is, indeed, the limit of a Riemann sum. Hence we obtain

\[
A = \int_{a}^{b} 2\pi f(x) \sqrt{1 + f'(x)^2} \, dx
\]
1) Geometric Series Test

\[ \sum_{n=m}^{\infty} c \cdot r^n = \begin{cases} \frac{Cr^m}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases} \]

\[ \text{Ex: } \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} \]

\[ \sum_{n=0}^{\infty} \frac{7^n}{5^n} \text{ diverges} \]

2) Nth-term Divergence Test

If \( a_n \to 0 \), then \( \sum_{n=1}^{\infty} a_n \) diverges.

\[ \text{Ex: } \sum_{n=1}^{\infty} \left(1 - \frac{2}{n}\right)^n \text{ diverges; } (1 - \frac{2}{n})^n \to e^{-2} \]

3) Integral Test

If \( f(x) \) is positive, decreasing, and continuous, then \( \sum_{n=1}^{\infty} f(n) \) converges \( \iff \int_1^{\infty} f(x) \, dx \)
also converges.
Ex. \[ \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \text{ diverges} \]
\[ \sum_{n=2}^{\infty} \frac{1}{n \ (n(n)^2)} \text{ converges} \]

4. **P-test**
\[ \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges } \iff \ p > 1 \]

Ex. \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges} \]
\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges} \]

5. **Direct Comparison Test (DCT)**
Suppose \( 0 \leq a_n \leq b_n \) for all \( n \).
If \( \sum b_n \) converges, then \( \sum a_n \) converges.
If \( \sum a_n \) diverges, then \( \sum b_n \) diverges.

Ex. \[ \sum_{n=1}^{\infty} \frac{1}{n \ln(n)} \text{ diverges, compare to } \sum_{n=1}^{\infty} \frac{1}{n} \]
\[ \sum_{n=1}^{\infty} \frac{1}{n^2+n} \text{ converges, compare to } \sum_{n=1}^{\infty} \frac{1}{n^2} \]
(6) Limit Comparison Test (LCT)

Suppose \( a_n \geq 0 \) and \( b_n > 0 \), for all \( n \).

Let \( L = \lim_{n \to \infty} \frac{a_n}{b_n} \).

(a) If \( 0 < L < \infty \), then \( \sum a_n \) and \( \sum b_n \) both converge or both diverge.

(b) If \( L = 0 \) and \( \sum b_n \) converges, then \( \sum a_n \) converges.

(c) If \( L = \infty \) and \( \sum b_n \) diverges, then \( \sum a_n \) diverges.

\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 3n + 1}} \text{ diverges, compare to } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \]

\[ \sum_{n=2}^{\infty} \frac{\ln(n)}{n^3} \text{ converges, compare to } \sum_{n=2}^{\infty} \frac{1}{n^2} \]

(7) Absolute Convergence Test

If \( \sum |a_n| \) converges, then \( \sum a_n \) converges.

\[ \sum_{n=1}^{\infty} \frac{\sin(n)}{n^2} \text{ converges; } \frac{\sin(n)}{n^2} \leq \frac{1}{n^2} \]
8. Alternating Series Test (AST)

If \( \{b_n\} \) is positive, decreasing, and has limit 0, then \( \sum (-1)^n b_n \) converges.

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \text{ converges}
\]

\( \downarrow \) AST Approximation Theorem

If \( \{b_n\} \) is as above, then

\[
\left| \sum_{n=1}^{N} (-1)^n b_n - \sum_{n=1}^{\infty} (-1)^n b_n \right| < b_{N+1}
\]

The error of the Nth partial sum is at most the size of the first omitted term.

9. Ratio Test

Suppose \( a_n \neq 0 \) and let \( p = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \)

(a) If \( p < 1 \), \( \sum a_n \) absolutely converges.
(b) If \( p > 1 \), \( \sum a_n \) diverges.
(c) If \( p = 1 \), test is inconclusive.

\( \downarrow \) Ex.

\[
\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \text{ converges, } p = \frac{1}{4}
\]
\[ \sum_{n=1}^{\infty} \frac{n^n}{n!} \text{ diverges, } \rho = e \]

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ ( } \rho = 1, \text{ test inconclusive) } \]

\[ \sum_{n=1}^{\infty} \frac{1}{n} \text{ ( } \rho = 1, \text{ test inconclusive) } \]

\[ \text{\textcircled{10} Root Test} \]

Let \( \rho = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \)

(a) If \( \rho < 1 \), \( \sum a_n \) absolutely converges.

(b) If \( \rho > 1 \), \( \sum a_n \) diverges.

(c) If \( \rho = 1 \), test is inconclusive.

\[ \sum_{n=1}^{\infty} (1 - \frac{2}{n})^n \text{ converges, } \rho = e^{-2} \]
Maclaurin and Taylor Series

1. Standard Maclaurin Series:
   \[
   \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \quad |x| < 1
   \]
   \[
   e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}
   \]
   \[
   \sin(x) = x - \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}
   \]
   \[
   \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}
   \]

2. Finding ROC and IOC of Power Series
   \[
   \sum_{n=0}^{\infty} a_n (x-c)^n
   \]
   (a) Use Ratio Test or Root Test to find that series converges for \(|x-c| < R\).
   The ROC is \(R\).
   (b) Check endpoints \(x = c+R\) and \(x = c-R\) for convergence separately.
Finding Taylor Series of \( f(x) \):

\[
T(x) = \sum_{n=0}^{\infty} a_n (x-c)^n
\]

(a) Use Cauchy formula \( a_n = \frac{f^{(n)}(c)}{n!} \)

OR

(b) Find \( T(x) \) from known power series, using substitution, differentiation, or integration as necessary.