1. Determine whether each of the following series converges or diverges.

(a) \[ \sum_{n=2}^{\infty} \sqrt{\frac{4n+1}{n^3-1}} \]

(b) \[ \sum_{n=1}^{\infty} \frac{(n+1)^5}{3^n} \]

(c) \[ \sum_{n=1}^{\infty} \frac{\sin(n)}{n^e+1} \]

(d) \[ \sum_{n=1}^{\infty} (-1)^n \sqrt{n+4} \]

\[ 5 \text{ pts} \]

Solution

(a) Observe the following:
\[ \sqrt{\frac{4n+1}{n^3-1}} > \sqrt{\frac{4n}{n^3-1}} > \sqrt{\frac{4n}{n^3}} = \frac{2}{n} > \frac{1}{n} \]

The series \[ \sum_{n=2}^{\infty} \frac{1}{n} \] diverges by \( p \)-test \((p = 1 \leq 1)\), and so the series \[ \sum_{n=2}^{\infty} \sqrt{\frac{4n+1}{n^3-1}} \] diverges by direct comparison test (DCT).

(b) We use Ratio Test.
\[ \rho = \lim_{n \to \infty} \left| \frac{(n+2)^5}{3^{n+1}} \cdot \frac{3^n}{(n+1)^5} \right| = \lim_{n \to \infty} \frac{1}{3} \cdot \frac{(n+2)^5}{(n+1)^5} = \frac{1}{3} \]

Since \( \rho < 1 \), the series converges by Ratio Test.

(c) Observe the following:
\[ \left| \frac{\sin(n)}{n^e+1} \right| = \frac{|\sin(n)|}{n^e+1} \leq \frac{1}{n^e+1} < \frac{1}{n^e} \]

The series \[ \sum_{n=1}^{\infty} \frac{1}{n^e} \] converges by \( p \)-test \((p = e > 1)\), and so the series \[ \sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^e+1} \right| \] converges by direct comparison test (DCT). Hence the series \[ \sum_{n=1}^{\infty} \frac{\sin(n)}{n^e+1} \] absolutely converges, hence converges.

(d) Note that \( \lim_{n \to \infty} (-1)^n \sqrt{n+4} \) does not exist, in particular, the limit is not 0. So the series diverges by the \( N \)th term divergence test.

\[ 5 \text{ pts} \]

2. Let \( S = \sum_{n=2}^{\infty} \frac{3^n + 3^{-n}}{4^n} \). If \( S \) converges, find its sum. If \( S \) diverges, prove that it diverges.

\[ 10 \text{ pts} \]
Solution
We write \( S \) as the sum of two convergent geometric series.

\[
S = \sum_{n=2}^{\infty} \frac{3^n}{4^n} + \sum_{n=2}^{\infty} \frac{3^{-n}}{4^n} = \sum_{n=2}^{\infty} \left( \frac{3}{4} \right)^n + \sum_{n=2}^{\infty} \left( \frac{1}{12} \right)^n
\]

The common ratio for the last two series are \( r = \frac{3}{4} \) and \( r = \frac{1}{12} \), respectively. Hence each series converges, and we may calculate \( S \) using the formula for a geometric series.

\[
S = \frac{(3/4)^2}{1 - 3/4} + \frac{(1/12)^2}{1 - 1/12} = \frac{9}{4} + \frac{1}{132} = \frac{149}{66}
\]

3. Find the value of \( \sum_{n=2}^{\infty} \frac{1}{(n+2)(n+3)} \).

Solution
We write the series as a telescoping series. The partial fraction decomposition of the summand is given by the following.

\[
\frac{1}{(n+2)(n+3)} = \frac{1}{n+2} - \frac{1}{n+3}
\]

Hence the \( N \)th partial sum is given by the following.

\[
S_N = \sum_{n=2}^{N} \left( \frac{1}{n+2} - \frac{1}{n+3} \right) = \left( \frac{1}{4} - \frac{1}{5} \right) + \left( \frac{1}{5} - \frac{1}{6} \right) + \cdots + \left( \frac{1}{N+2} - \frac{1}{N+3} \right) = \frac{1}{4} - \frac{1}{N+3}
\]

By definition, \( S = \lim_{N \to \infty} \left( \frac{1}{4} - \frac{1}{N+3} \right) = \frac{1}{4} \).

4. Let \( S = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}} \). Find the minimum value of \( N \), such that \( S_N \) (the \( N \)th partial sum of \( S \)) is guaranteed to approximate \( S \) with an error no greater than \( 1/7 \).

Solution
Let \( a_n = \frac{1}{\sqrt{n+3}} \). Then \( \{a_n\} \) is positive, is decreasing, and has limit 0. So \( a_n \) satisfies the hypotheses of the alternating series test. Thus we may use the alternating series approximation theorem. Recall that the error incurred by the truncated sum is at most the size of the first omitted term. So we seek the minimum value of \( N \) such that \( a_{N+1} < 1/7 \), or \( 1/\sqrt{N+4} < 1/7 \). Hence \( N > 45 \), and so the minimum value of \( N \) is 46.

5. Determine whether the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)^2}{n^3} \) converges absolutely, converges conditionally, or diverges.
Solution

Let \( a_n = \frac{(n+2)^2}{n^3} \), and observe the following.

\[
\frac{(n+2)^2}{n^3} > \frac{n^2}{n^3} = \frac{1}{n}
\]

The series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges by \( p \)-test (\( p = 1 \leq 1 \)), and so the series \( \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}(n+2)^2}{n^3} \right| = \sum_{n=1}^{\infty} \frac{(n+2)^2}{n^3} \) diverges by the direct comparison test (DCT). (So the original series does not absolutely converge.)

Note that \( \{a_n\} \) is positive, is decreasing, and has limit 0. For the “decreasing” part, observe that

\[
\frac{d}{dx} \left( \frac{(x+2)^2}{x^3} \right) = \frac{-(x+2)(x+6)}{x^4} < 0
\]

Hence the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)^2}{n^3} \) converges by alternating series test (AST).

We conclude that the given series converges conditionally.

10 pts 6. Find the interval of convergence of \( \sum_{n=1}^{\infty} \frac{n(x+3)^n}{3^n} \).

Solution

We start with Ratio Test.

\[
\rho = \lim_{n \to \infty} \left| \frac{(n+1)(x+3)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n(x+3)^n} \right| = \lim_{n \to \infty} \frac{|x+3|}{3} \cdot \frac{n+1}{n} = \frac{|x+3|}{3}
\]

By Ratio Test, the series converges for \( |x+3|/3 < 1 \) and diverges for \( |x+3|/3 > 1 \). Hence the radius of convergence is \( R = 3 \) and the interval of convergence is \((-6,0)\), possibly with one or both endpoints.

Now we check each endpoint. For \( x = -6 \), we observe that the series \( \sum_{n=1}^{\infty} (-1)^n n \) diverges by the \( N \)th term divergence test since \( \lim_{n \to \infty} (-1)^n n \) does not exist, in particular, the limit is not 0. For \( x = 0 \), we observe that the series \( \sum_{n=1}^{\infty} n \) diverges by the \( N \)th term divergence test since \( \lim_{n \to \infty} n = \infty \neq 0 \).

Hence the interval of convergence is \((-6,0)\).

7. Let \( f(x) = \frac{1}{3 - x^2} \) and let \( g(x) = \frac{x}{(3 - x^2)^2} \).

Note: For each part, you must give your answer in summation (\( \Sigma \)) notation for full credit, but for partial credit you may write the first four nonzero terms of the series.
(a) Use a geometric series expansion to find a power series for \( f(x) \) with center \( c = 0 \).

(b) Use your answer from part (a) to find a power series for \( g(x) \).

**Solution**

(a) Recall that \( \frac{1}{1-u} = \sum_{n=0}^{\infty} u^n \) (for \( |u| < 1 \)). Hence we have the following.

\[
\frac{1}{3-x^2} = \frac{1}{3} \cdot \frac{1}{1 - \frac{x^2}{3}} = \frac{1}{3} \sum_{n=0}^{\infty} \left( \frac{x^2}{3} \right)^n = \sum_{n=0}^{\infty} \frac{x^{2n}}{3^{n+1}}
\]

(b) Observe that \( g(x) = \frac{1}{2} f'(x) \). Hence we have the following.

\[
\frac{1}{3-x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{3^{n+1}}
\]

\[
\frac{d}{dx} \left( \frac{1}{3-x^2} \right) = \sum_{n=0}^{\infty} \frac{d}{dx} \left( \frac{x^{2n}}{3^{n+1}} \right)
\]

\[
\frac{2x}{(3-x^2)^2} = \sum_{n=0}^{\infty} \frac{2n}{3^{n+1}} x^{2n-1}
\]

\[
\frac{x}{(3-x^2)^2} = \sum_{n=0}^{\infty} \frac{n}{3^{n+1}} x^{2n-1}
\]

8. Find the first four nonzero terms of the Maclaurin series of \( (1 + x)^{2/3} \).

**Solution**

We use the Cauchy formula.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(0) )</th>
<th>( a_n = \frac{f^{(n)}(0)}{n!} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( (1 + x)^{2/3} )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{2}{3} (1 + x)^{-1/3} )</td>
<td>( \frac{2}{3} )</td>
<td>( \frac{2}{3} )</td>
</tr>
<tr>
<td>2</td>
<td>( -\frac{2}{9} (1 + x)^{-4/3} )</td>
<td>( -\frac{2}{9} )</td>
<td>( -\frac{1}{9} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{8}{27} (1 + x)^{-7/3} )</td>
<td>( \frac{8}{27} )</td>
<td>( \frac{4}{81} )</td>
</tr>
</tbody>
</table>

Hence the first four non-zero terms of the Maclaurin series are

\[ 1 + \frac{2}{3} x - \frac{1}{9} x^2 + \frac{4}{81} x^3 + \cdots \]
9. Evaluate \[ \int_{3}^{\infty} \frac{2x}{(x^2 + 3)^2} \, dx. \]

**Solution**

We write the integral as a limit of Riemann integrals and use the substitution \( u = x^2 + 3 \) (whence \( du = 2x \, dx \)).

\[
\int_{3}^{\infty} \frac{2x}{(x^2 + 3)^2} \, dx = \lim_{R \to \infty} \int_{3}^{R} \frac{2x}{(x^2 + 3)^2} \, dx = \lim_{R \to \infty} \int_{12}^{R^2 + 3} \frac{1}{u^2} \, du
\]

\[
= \lim_{R \to \infty} \left( \frac{-1}{u} \right)_{12}^{R^2 + 3} = \lim_{R \to \infty} \left( \frac{1}{12} - \frac{1}{R^2 + 3} \right) = \frac{1}{12}
\]