1. Consider the function 
\[ f(x) = 4e^{-x^2/8} \]
Find where \( f \) is concave down and find where \( f \) is concave up. Then determine the coordinates of all inflection points. 

**Your intervals should be as inclusive as possible. Write “does not exist” for your answer if appropriate.**

**Solution**
The first two derivatives of \( f \) are 
\[ f'(x) = -xe^{-x^2/8}, \quad f''(x) = \frac{1}{4}(x^2 - 4)e^{-x^2/8} \]
Since \( f \) is twice-differentiable on its domain, the only second-order critical numbers are solutions to \( f''(x) = 0 \).
\[ \frac{1}{4}(x^2 - 4)e^{-x^2/8} = 0 \Rightarrow x = -2, 2 \]
We make a sign chart for \( f''(x) \). Note that \( \frac{1}{4}e^{-x^2/8} \) is positive for all \( x \). So we need only test the sign of \( x^2 - 4 \).

<table>
<thead>
<tr>
<th>interval</th>
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<th>sign of ( f'' )</th>
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<tbody>
<tr>
<td>((-\infty, -2))</td>
<td>(f''(-3) = 5)</td>
<td>⊕</td>
<td>concave up</td>
</tr>
<tr>
<td>((-2, 2))</td>
<td>(f''(0) = -4)</td>
<td>⊖</td>
<td>concave down</td>
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<td>((2, \infty))</td>
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Hence \( f \) is concave down on the interval \([-2, 2]\) and concave up on the intervals \((-\infty, -2]\) and \([2, \infty)\). There are points of inflection at \((-2, 4e^{-1/2})\) and \((2, 4e^{-1/2})\).

2. Let \( f(x) = 12x^{1/3} - x \). Note: The domain of \( f \) is \((-\infty, \infty)\).

(a) Calculate all critical numbers of \( f \). **For each number you find, you should clearly indicate in your work why your answer is a critical number.**

(b) What are the global extreme values of \( f \) on the interval \([-1, 27]\)?

**Solution**
(a) Note that \( f \) is continuous for all \( x \). So the critical numbers of \( f \) are those values of \( x \) for which either \( f'(x) \) does not exist or \( f'(x) = 0 \). We first note that \( x^{1/3} \) is not differentiable at \( x = 0 \), hence \( x = 0 \) is also a critical number
of $f$. The derivative is

$$f'(x) = 4x^{-2/3} - 1$$

Solving the equation $f'(x) = 0$ gives us the solutions $x = -8$ and $x = 8$. So, in summary, $f$ has three critical numbers: $x = 0$, $x = -8$, and $x = 8$.

(b) Since $f$ is continuous on the closed and bounded interval $[-1, 27]$, the extreme values of $f$ exist and must be critical values. Checking the critical numbers and endpoints, we find that $f(-1) = -11$, $f(0) = 0$, $f(8) = 16$, and $f(27) = 9$. Hence the global minimum value of $f$ on $[-1, 27]$ is $-11$ and the global maximum value is $16$.

### 3. Consider the function

$$f(x) = \frac{1}{x^3 - 6x^2}$$

First find all vertical asymptotes of $f$. Then find where $f$ is decreasing and find where $f$ is increasing. Finally determine the $x$-coordinates of all local extrema of $f$. 

**Your intervals should be as inclusive as possible. Write “does not exist” for your answer if appropriate.**

#### Solution

The domain of $f$ is all real numbers except where $x^3 - 6x^2 = 0$, or $x^2(x - 6) = 0$. Hence the only numbers not in the domain of $f$ are $x = 0$ and $x = 6$. Since $f$ is algebraic, we know that $f$ is continuous on its domain, hence the only candidates for vertical asymptotes are the lines $x = 0$ and $x = 6$. Direct substitution of either $x = 0$ or $x = 6$ gives the undefined expression $\frac{1}{0}$. So we know that all corresponding one-sided limits at $x = 0$ and $x = 6$ must be infinite. So both lines $x = 0$ and $x = 6$ are, indeed, vertical asymptotes.

For intervals of increase and local extrema, we examine the first derivative.

$$f'(x) = -\frac{3x^2 - 12x}{(x^3 - 6x^2)^2} = -\frac{3x(x - 4)}{(x^3 - 6x^2)^2}$$

Since $f$ is differentiable on its domain, the only first-order critical numbers are solutions to $f'(x) = 0$.

$$-\frac{3x(x - 4)}{(x^3 - 6x^2)^2} = 0 \implies x = 4$$

(Note that $x = 0$ is not a solution.) We make a sign chart for $f'(x)$. Recall that since $x = 0$ and $x = 6$ are not in the domain of $f$, we must include those numbers on our sign chart.
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<td>$-$</td>
<td>decreasing</td>
</tr>
<tr>
<td>$(0, 4)$</td>
<td>$f'(1) = +\infty$</td>
<td>$+$</td>
<td>increasing</td>
</tr>
<tr>
<td>$(4, 6)$</td>
<td>$f'(5) = -\infty$</td>
<td>$-$</td>
<td>decreasing</td>
</tr>
<tr>
<td>$(6, \infty)$</td>
<td>$f'(7) = -\infty$</td>
<td>$-$</td>
<td>decreasing</td>
</tr>
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Hence $f$ is decreasing on the intervals $(-\infty, 0)$, $[4, 6)$, and $(6, \infty)$; and $f$ is increasing on the interval $(0, 4]$. There is no local minimum, but there is a local maximum at $x = 4$.

4. For each limit, calculate the value or show that it does not exist. If the limit is $+\infty$ or $-\infty$, that should be your answer instead of “does not exist”. Show all work.

(a) \[ \lim_{x \to 0} \frac{\ln(1 + 9x) - 9x}{1 - \cos(5x)} \]

(b) \[ \lim_{x \to 0^+} (1 - 8x)^{3/x} \]

**Solution**

(a) Use L’Hospital’s Rule twice.

\[
\lim_{x \to 0} \left( \frac{\ln(1 + 9x) - 9x}{1 - \cos(5x)} \right) = \lim_{x \to 0} \left( \frac{9}{5 \sin(5x)} \right) = \lim_{x \to 0} \left( \frac{81}{25 \cos(5x)} \right) = \frac{81}{25}
\]

(b) Direct substitution of $x = 0^+$ gives the indeterminate exponent $1^\infty$. So we let $L$ be the desired limit and we consider $\ln(L)$. What follows is a standard application of L’Hospital’s Rule.

\[
\ln(L) = \lim_{x \to 0^+} \ln \left( (1 - 8x)^{3/x} \right) = \lim_{x \to 0^+} \left( \frac{3 \ln(1 - 8x)}{\frac{x^{24}}{1 - 8x}} \right) = \lim_{x \to 0^+} \left( \frac{-24}{x^{24}} \right) = -24
\]

So $\ln(L) = -24$, whence $L = e^{-24}$. 

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5. For all parts of this question, let \( f(x) = \sqrt{40x + 20} - x^2 \).

(a) Let \( L(x) \) be the linearization of \( f(x) \) at \( x = 2 \). Find \( L(x) \).

(b) Use a linear approximation to estimate the value of \( f(2.04) \).

(c) Consider using Newton’s method to find an approximate solution to the equation \( f(x) = 0 \). Find \( x_1 \) if the initial guess is \( x_0 = 2 \).

Solution

(a) First note that \( f(2) = 6 \). We also have that

\[
f'(x) = \frac{20}{\sqrt{40x + 20}} - 2x
\]

whence \( f'(2) = -2 \). Hence

\[
L(x) = 6 - 2(x - 2) = 10 - 2x
\]

(b) Our linear approximation implies that \( f(2.04) \approx L(2.04) = 5.92 \).

(c) Since \( L(x) \) is the tangent line to \( y = f(x) \) at \( x = x_0 \), the next iterate \( x_1 \) is the solution to \( L(x) = 0 \), which is \( x_1 = 5 \). Alternatively, we may use the formula

\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{6}{-2} = 5
\]

6. The total surface area of a cube is changing at a rate of 12 in\(^2\)/s when the length of one of the sides is 10 in. At what rate is the volume of the cube changing at that time?

You must include correct units as part of your answer.

Solution

Let \( x \) be the side length of the cube. Then the total surface area and volume of the cube are

\[
S = 6x^2 \quad , \quad V = x^3
\]

Differentiating with respect to time \( t \) gives

\[
\frac{dS}{dt} = 12x \frac{dx}{dt} \quad , \quad \frac{dV}{dt} = 3x^2 \frac{dx}{dt}
\]

These four equations hold for all time. Now we substitute the information relevant to the specific time, i.e., \( x = 10 \) and \( \frac{dS}{dt} = 12 \).

\[
S = 600 \quad , \quad V = 1000 \quad , \quad 12 = 120 \frac{dx}{dt} \quad , \quad \frac{dV}{dt} = 300 \frac{dx}{dt}
\]
Solving for \( \frac{dx}{dt} \) in the third equation gives \( \frac{dx}{dt} = \frac{12}{120} = \frac{1}{10} \). Substituting into the fourth equation gives

\[
\frac{dV}{dt} = 300 \cdot \frac{1}{10} = 30
\]

Hence the volume of the cube is increasing at a rate of 30 in\(^3\)/sec.

7. Find the maximum possible area of a rectangle inscribed in the region below the graph of \( y = \frac{4}{(x + 2)^2} \) and in the first quadrant.

**You must demonstrate that your answer really is the maximum area!**

**Solution**

Let the coordinates of the upper right vertex of the rectangle be \((x, y)\). Then the area of the rectangle is \( A(x, y) = xy \), and we want to maximize \( A \) subject to the constraint that the upper right vertex lies on the given curve. (That is, \( y = \frac{4}{(x+2)^2} \).) Hence our goal is to find the maximum value of the function

\[
f(x) = \frac{4x}{(x+2)^2}
\]

on the interval \([0, \infty)\). The function \( f \) is differentiable on \((0, \infty)\), thus the only critical numbers are solutions to \( f'(x) = 0 \).

\[
0 = f'(x) = \frac{-4(x - 2)}{(x + 2)^3} \implies x = 2
\]

By way of the first derivative test, we note that \( f'(1) = \frac{4}{27} > 0 \) and \( f'(3) = -\frac{4}{125} < 0 \). Hence \( f(x) \) is increasing on \([0, 2] \) and decreasing on \([2, \infty) \). This implies that \( f(2) \) is, indeed, the maximum value of \( f \). The maximum area is thus \( f(2) = \frac{1}{2} \).