1. A camera is located 5 feet from a straight wire along which a bead is moving at 6 feet per second. The camera automatically turns so that it is pointed at the bead at all times. How fast is the camera turning 2 seconds after the bead passes closest to the camera? You must express your final answer as an exact rational number and indicate its units.

Solution
Suppose the bead travels along the $x$-axis. Let $O = (0,0)$ be the origin and let $C = (0,5)$ be the coordinates of the camera. Let $P = (x,0)$ be the coordinates of the bead and let $\theta = \angle OCP$. Then $x = 5 \tan(\theta)$, whence $\frac{dx}{dt} = 5 \sec(\theta)^2 \frac{d\theta}{dt}$. The bead moves to the right at a rate of 6 ft/s ($\frac{dx}{dt} = 6$), and so the bead is at $Q = (12,0)$ 2 seconds after it has passed closest to the camera. Hence at this time we have

$$12 = 5 \tan(\theta)$$
$$6 = 5 \sec(\theta)^2 \frac{d\theta}{dt}$$

Note that in $\triangle OQC$, the hypotenuse has length $\sqrt{12^2 + 5^2} = 13$ and the side adjacent to $\theta$ has length 5. Hence $\sec(\theta) = \frac{13}{5}$. Substitution into the second equation then gives us

$$6 = 5 \left(\frac{13}{5}\right)^2 \frac{d\theta}{dt} \implies \frac{d\theta}{dt} = \frac{30}{169}$$

(The units are radians per second.)

2. Calculate each of the following limits or show it does not exist. Show all work.

(a) $\lim_{x \to 0} \frac{\cos(4x)^{6/x^2}}{x^2}$

(b) $\lim_{x \to 1} \frac{xe^{4x} + 4e^4 - 5e^4x}{(x-1)^2}$

Solution
(a) Substitution of $x = 0$ gives the indeterminate form $1^\infty$. Let $L$ be the desired limit and consider $\ln(L)$. Then we have

$$\ln(L) = \ln \left( \lim_{x \to 0} \cos(4x)^{6/x^2} \right) = \lim_{x \to 0} \ln \left( \cos(4x)^{6/x^2} \right) = \lim_{x \to 0} \left( \frac{6 \ln(\cos(4x))}{x^2} \right)$$

We have used continuity of the logarithm and logarithm identities. Substitution of $x = 0$ now gives the indeterminate form $\frac{0}{0}$, whence we may use L’Hospital’s
Rule (and for any subsequent indeterminate forms of $\frac{0}{0}$).

$$
\lim_{x \to 0} \left( \frac{6 \ln(\cos(4x))}{x^2} \right) = \frac{H}{\lim_{x \to 0} \left( \frac{6 \cdot \frac{1}{\cos(4x)} \cdot (-4 \sin(4x))}{2x} \right)}
$$

$$
= \lim_{x \to 0} \left( -12 \cdot \frac{\tan(4x)}{x} \right) = \frac{H}{\lim_{x \to 0} \left( -12 \cdot \frac{4 \sec(4x)^2}{1} \right)} = -48
$$

Hence $\ln(L) = -48$, and so $L = e^{-48}$.

(b) Substitution of $x = 0$ gives the indeterminate form $\frac{0}{0}$, whence we may use L'Hospital's Rule (and for any subsequent indeterminate forms of $\frac{0}{0}$).

$$
\lim_{x \to 1} \left( \frac{xe^{4x} + 4e^4 - 5e^4x}{(x - 1)^2} \right) = \frac{H}{\lim_{x \to 1} \left( \frac{4xe^{4x} + e^{4x} - 5e^4}{2(x - 1)} \right)}
$$

$$
= \lim_{x \to 0} \left( \frac{16xe^{4x} + 8e^{4x}}{2} \right) = \frac{24e^4}{2} = 12e^4
$$

3. Find the minimum and maximum values of

$$
f(x) = x(\ln(x) - 4)^2
$$
on the interval $[e^{-3}, e^3]$.

**Solution**

The function $f$ is continuous on the given closed interval and differentiable on the interior. Hence the extreme values can occur only at interior points where $f'(x) = 0$ or at the interval endpoints.

$$
f'(x) = 2x(\ln(x) - 4) \cdot \frac{1}{x} + (\ln(x) - 4)^2 = (\ln(x) - 4)(\ln(x) - 2)
$$

Hence the only critical points are $x = e^2$ and $x = e^4$ (not in the interval). We now check the function values at the candidate points.

$$
f(e^{-3}) = e^{-3}(-3 - 4)^2 = \frac{49}{e^6}$$

$$
f(e^2) = e^2(2 - 4)^2 = 4e^2$$

$$
f(e^3) = e^3(3 - 4)^2 = e^3$$

Since $e > 2$, it follows that $e^3 > 8$, or $e^3 > 7$. Hence $f(e^{-3}) = \frac{49}{e^6} < \frac{49}{7} = 7 < e^3 = f(e^3)$. Since $e < 4$, it follows that $f(e^3) = e^3 = e \cdot e^2 < 4 \cdot e^2 = f(e^2)$. Hence $f(e^{-3}) < f(e^3) < f(e^2)$. So the maximum value of $f$ is $4e^2$ and the minimum value of $f$ is $\frac{49}{e^6}$.
4. Consider the function \( f(x) = x^2 + 5x \) on the interval \([0, 4]\).

(a) Calculate \( L_4 \). Your answer should be written as an exact rational number.

(b) Find an explicit formula for \( R_N \) in terms of \( N \). Then use your answer to calculate the area under the graph of \( y = f(x) \) over the interval \([0, 4]\). The following formulas may be useful.

\[
\sum_{k=1}^{N} k = \frac{N(N+1)}{2}, \quad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}, \quad \sum_{k=1}^{N} k^3 = \frac{N^2(N+1)^2}{4}
\]

Solution

(a) We find that \( \Delta x = \frac{4-0}{4} = 1 \), and so the subinterval endpoint values are \( x_k = 0, 1, 2, 3, 4 \) for \( k = 0, \ldots, 4 \). The left endpoint values are \( x_k = 0, 1, 2, 3 \) for \( k = 0, \ldots, 3 \), and the corresponding \( y \)-values are \( y_k = 0, 6, 14, 24 \). Hence we have

\[
L_4 = \Delta x (y_0 + y_1 + y_2 + y_3) = (1)(0 + 6 + 14 + 24) = 44
\]

(b) For general \( N \), we have \( \Delta x = \frac{4}{N} \). The right endpoint values are \( x_k = 0 + k \Delta x = \frac{4k}{N} \) for \( k = 1, \ldots, N \). Hence we have

\[
R_N = \Delta x \sum_{k=1}^{N} f(x_k) = \frac{4}{N} \sum_{k=1}^{N} \left( \left( \frac{4k}{N} \right)^2 + \frac{4}{N} \right) = \frac{4}{N} \sum_{k=1}^{N} \left( \frac{16k^2}{N^2} + \frac{20k}{N} \right)
\]

\[
= \frac{4}{N} \left( \sum_{k=1}^{N} \frac{16k^2}{N^2} + \sum_{k=1}^{N} \frac{20k}{N} \right)
\]

\[
= \frac{4}{N} \left( \frac{16}{N^2} \sum_{k=1}^{N} k^2 + \frac{20}{N} \sum_{k=1}^{N} k \right)
\]

\[
= \frac{4}{N} \left( \frac{16}{N^2} \cdot \frac{N(N+1)(2N+1)}{6} + \frac{20}{N} \cdot \frac{N(N+1)}{2} \right)
\]

\[
= \frac{32(N+1)(2N+1)}{3N^2} + \frac{40(N+1)}{N} = \frac{32}{3} (1 + \frac{1}{N}) (2 + \frac{1}{N}) + 40 \left( 1 + \frac{1}{N} \right)
\]

By definition, the area under the graph is

\[
\lim_{N \to \infty} R_N = \lim_{N \to \infty} \left( \frac{32}{3} (1 + \frac{1}{N}) (2 + \frac{1}{N}) + 40 \left( 1 + \frac{1}{N} \right) \right) = \frac{32}{3} (1)(2)+40(1) = \frac{184}{3}
\]

5. The parts of this question are independent.

(a) Complete the statement of the mean value theorem (MVT).

Suppose \( f(x) \) is _______ for all \( x \) in \([a, b]\) and _______ for all \( x \) in _______. Then there exists a number \( c \) in the interval _______ such that the following equation is satisfied (write the equation in the space below):

\[ \]
(b) For each of the following parts, determine whether the MVT applies. If MVT does not apply, explain why. If MVT does apply, find all values of \( c \) guaranteed to exist by the MVT.

(i) \( f(x) = (x^2 - 2x)^{1/3} \) on \([-2, 4]\)

(ii) \( f(x) = \frac{x + 4}{x - 4} \) on \([-3, 3]\)

Solution

(a) Suppose \( f(x) \) is continuous for all \( x \) in \([a, b]\) and differentiable for all \( x \) in \((a, b)\). Then there exists a number \( c \) in the interval \((a, b)\) such that the following equation is satisfied:

\[
\frac{f(b) - f(a)}{b - a} = f'(c)
\]

(b) (i) The function \( g(x) = x^{1/3} \) is not differentiable at \( x = 0 \), hence \( f(x) \) is not differentiable whenever \( x^2 - 2x = 0 \), or at \( x = 0 \) and \( x = 2 \). These values of \( x \) are in the interval \([-2, 4]\), hence the MVT does not apply.

(ii) The equation guaranteed to have a solution by the MVT is given by

\[
-\frac{8}{(c - 4)^2} = f'(c) = \frac{f(3) - f(-3)}{3 - (-3)} = \frac{-7 - \frac{-1}{7}}{6} = -\frac{8}{7}
\]

Hence \((c - 4)^2 = 7\), or \( c = 4 \pm \sqrt{7} \). However, the value \( 4 + \sqrt{7} \) is not in the given interval. So the only value of \( c \) guaranteed to exist by the MVT is \( c = 4 - \sqrt{7} \).

6. Use a linear approximation to estimate the value of \( \sqrt{33} \). Express your answer as an exact rational number.

Solution

If \( \Delta x \) is small, then we may use the formula

\[
f(a + \Delta x) \approx f(a) + f'(a)\Delta x
\]

Let \( f(x) = \sqrt{x} \), \( a = 36 \), and \( \Delta x = -3 \). Then in this language, our formula is

\[
f(33) \approx f(36) + f'(36)(-3)
\]

Observe that \( f(36) = 6 \) and \( f'(36) = \frac{1}{2\sqrt{36}} = \frac{1}{12} \). Hence we have

\[
\sqrt{33} \approx 6 + \frac{1}{12}(-3) = \frac{23}{4}
\]
7. A rectangular storage container must have a volume of 10 ft$^3$. The length of its base is twice the width. Material for the top and base costs $12/ft^2$ and material for the sides costs $6/ft^2$. Find the dimensions of the cheapest container (length, width, and height). You must include correct units as part of your answer.

You must give a full justification for your answer using methods taught in this course. You must also demonstrate that your answer really does give the minimum cost.

Solution
Let $x$, $y$, and $z$ be the length, width, and height of the box, respectively. The cost of the box in terms of these variables is

$$C = 12(2xy) + 6(2xz + 2yz) = 24xy + 12xz + 12yz$$

We are given that $y = 2x$ and $10 = xyz$ (volume equation). Substitution of $y = 2x$ into $10 = xyz$ gives $z = \frac{5}{x^2}$. Substitution into our cost function gives us cost in terms of $x$ only.

$$C(x) = 24x(2x) + 12x\left(\frac{5}{x^2}\right) + 12(2x)\left(\frac{5}{x^2}\right) = 48x^2 + \frac{180}{x}$$

Our goal is to find the minimum value of $C(x)$ on the interval $x \in (0, \infty)$. The function $C$ is differentiable on this interval, hence the only critical points are those values of $x$ such that $C'(x) = 0$.

$$0 = C'(x) = 96x - \frac{180}{x^2}$$

Solving for $x$ gives

$$x = \left(\frac{180}{96}\right)^{1/3}$$

The width and height of the box are

$$y = 2x = 2\left(\frac{180}{96}\right)^{1/3}, \quad z = 5x^{-2} = 5\left(\frac{180}{96}\right)^{-2/3}$$

(The values $x$, $y$, and $z$ all have units of feet.) Now observe that the second derivative of the cost is

$$C''(x) = 96 + \frac{360}{x^3}$$

which is strictly positive for all $x > 0$. Hence the graph of $C(x)$ is concave up on the entire interval $(0, \infty)$. This means that the only critical point we found must give a global minimum of $C(x)$. 

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8. Consider the function \( f \) and its derivatives given below.

\[
\begin{align*}
  f(x) &= \frac{1}{(x + 4)^2(x - 6)^2}, \\
  f'(x) &= \frac{-2(2x - 2)}{(x + 4)^3(x - 6)^3}, \\
  f''(x) &= \frac{10(2x^2 - 4x + 12)}{(x + 4)^4(x - 6)^4}
\end{align*}
\]

Show all work for all of the following parts!

(a) Find the domain of \( f \).

(b) Calculate all \( x \)- and \( y \)-intercepts of \( f \).

(c) Determine all vertical and horizontal asymptotes of \( f \).

(d) Find where \( f \) is increasing and find where \( f \) is decreasing. Then calculate the \( x \)-coordinates of all local extrema, classifying each as either a local minimum or a local maximum.

(e) Find where \( f \) is concave up and find where \( f \) is concave down. Then calculate the \( x \)-coordinates of all inflection points.

(f) Sketch the graph of \( y = f(x) \) on the provided grid using your previous answers. Label each asymptote by its equation. Label each transition point by its \( x \)-coordinates and as “rel. min”, “rel. max”, or “infl. pt.”, as appropriate.

Make sure to draw and label axes. Your graph need not to be to scale, but it must have the correct shape.

Solution

(a) The domain is \((-\infty, -4) \cup (-4, 6) \cup (6, \infty)\).

(b) Note that the equation \( f(x) = 0 \) has no solution, hence there are no \( x \)-intercepts. The \( y \)-intercept is \((0, \frac{1}{24})\).

(c) Note that \( f \) is continuous on its domain. Hence the only candidate vertical asymptotes are \( x = -4 \) and \( x = 6 \). Substitution of either \( x = -4 \) or \( x = 6 \) into \( f(x) \) gives the expression \( \frac{1}{6} \) (i.e., a non-zero number divided by zero). We know that this means that all one-sided limits are infinite. Hence \( x = -4 \) and \( x = 6 \) are the vertical asymptotes.

Note that \( \lim_{x \to \pm \infty} f(x) = 0 \), hence the only horizontal asymptote is \( y = 0 \).

(d) The first derivative exists everywhere in the domain and vanishes when \( x = 1 \). Now we calculate a sign chart for \( f' \). Subintervals are determined by the points \( x = -4, 1, 6 \) because these are the only points where \( f \) either is discontinuous...
or has a critical point.

\[
x < -4 : \quad f'(-5) = \frac{(-)(-)}{(-)(-)} > 0
\]
\[
-4 < x < 1 : \quad f'(0) = \frac{(-)(-)}{(+)(-)} < 0
\]
\[
1 < x < 6 : \quad f'(2) = \frac{(-)(+)}{(+)(-)} > 0
\]
\[
6 < x : \quad f'(7) = \frac{(-)(+)}{(+) (+)} < 0
\]

Hence we find the following about the function \( f \).

\[
f \text{ decreasing on: } (-4, 1), (6, \infty)
\]
\[
f \text{ increasing on: } (-\infty, -4), (1, 6)
\]
\[
\text{local minimum value at: } x = 1
\]
\[
\text{local maximum value at: } \text{none}
\]

(e) The second derivative exists everywhere in the domain and vanishes nowhere. (The discriminant of \( 2x^2 - 4x + 12 \) is \( \Delta = -80 < 0 \). Hence there are no solutions to \( f''(x) = 0 \).) Observe that \( 2x^2 - 4x + 12 \) is never equal to 0 and is a parabola that opens upward, hence \( 2x^2 - 4x + 12 > 0 \) always. Also observe that \( (x + 4)^2(x - 6)^4 \geq 0 \) always. Hence \( f''(x) > 0 \) on its entire domain. Hence we find the following about the function \( f \).

\[
f \text{ concave down on: } \emptyset
\]
\[
f \text{ concave up on: } (-\infty, -4), (-4, 6), (6, \infty)
\]
\[
\text{inflection points at: } \text{none}
\]

(f) Using the previous solutions, we have the following sketch.