1. Find an equation of the line tangent to the graph of the equation \(2x^2 - xy + 5y^2 = 24\) at the point \((-1, 2)\).

**Solution**
Implicitly differentiating with respect to \(x\) gives
\[
4x - xy' - y + 10yy' = 0
\]
Substituting \(x = -1\) and \(y = 2\) gives
\[
4(-1) - (-1)y' - 2 + 10(2)y' = 0
\]
Solving for \(y'\) gives \(y' = \frac{6}{21} = \frac{2}{7}\). Hence an equation of the tangent line is
\[
y - 2 = \frac{2}{7}(x - (-1))
\]

2. Find equations of all horizontal asymptotes of the function
\[
f(x) = \frac{\sqrt{25x^2 - 6x}}{3x + 2}.
\]

**Solution**
We must compute the limit of \(f(x)\) as \(x \to \pm \infty\). First we do some algebra by factoring out the highest power in numerator and denominator.
\[
\frac{\sqrt{25x^2 - 6x}}{3x + 2} = \frac{\sqrt{x^2 \left( \frac{25}{x} - \frac{6}{x} \right)}}{x \left( 3 + \frac{2}{x} \right)} = \frac{|x| \cdot \sqrt{25 - \frac{6}{x}}}{x \cdot 3 + \frac{2}{x}}
\]
Now observe
\[
\lim_{x \to \pm \infty} \frac{25 - \frac{6}{x}}{3 + \frac{2}{x}} = \frac{25 - 0}{3 + 0} = \frac{5}{3}
\]
So now we have
\[
\lim_{x \to +\infty} f(x) = \frac{5}{3} \cdot \lim_{x \to +\infty} \left( \frac{|x|}{x} \right) = \frac{5}{3} \cdot \lim_{x \to +\infty} \left( \frac{x}{x} \right) = \frac{5}{3}
\]
\[
\lim_{x \to -\infty} f(x) = \frac{5}{3} \cdot \lim_{x \to -\infty} \left( \frac{|x|}{x} \right) = \frac{5}{3} \cdot \lim_{x \to -\infty} \left( \frac{-x}{x} \right) = -\frac{5}{3}
\]
Hence the horizontal asymptotes are \(y = \frac{5}{3}\) and \(y = -\frac{5}{3}\).
3. Consider the claim that \( \lim_{x \to 1} (x^2 - 2x) = -1 \).

(a) Write down the precise definition of limit (using \( \epsilon \) and \( \delta \), as it applies to this particular limit.

(b) Using the precise definition of limit, prove that the claim is true.

Solution

(a) The claim is true if for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( 0 < |x - 1| < \delta \), then \( |x^2 - 2x - (-1)| < \epsilon \).

(b) Fix \( \epsilon > 0 \). Put \( \delta = \sqrt{\epsilon} \) and suppose \( 0 < |x - 1| < \delta \). Then

\[
|x^2 - 2x - (-1)| = |x^2 - 2x + 1| = |(x - 1)^2| = |x - 1|^2 < \delta^2 = \epsilon
\]

Hence the claim is true.

4. Find the values of the constants \( a \) and \( b \) so that the following function is continuous for all \( x \). If this is not possible, explain why.

\[
f(x) = \begin{cases} 
ax + b & , \ x < 1 \\
-2 & , \ x = 1 \\
3 \sqrt{x} + b & , \ x > 1 
\end{cases}
\]

You must give a full, clear justification for your answer. You must use proper methods taught in this course.

Solution

The first two “pieces” of \( f(x) \) are continuous for all \( x \) regardless of the values of \( a \) and \( b \) since polynomials are continuous for all \( x \). The “piece” \( 3 \sqrt{x} + b \) is continuous regardless of the value of \( b \) as long as \( x \geq 0 \). Hence each piece is continuous on each of its “pieces” separately on the respective intervals. We need only force continuity at \( x = 1 \) to guarantee \( f \) is continuous for all \( x \). Hence we must choose \( a \) and \( b \) such that

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) \\
\lim_{x \to 1^-} (ax + b) = \lim_{x \to 1^+} (3 \sqrt{x} + b) = -2 \\
a + b = 3 + b = -2
\]

Hence \( b = -5 \) and \( a = 3 \).
5. On the set of axes provided below, sketch the graph of a function \( f(x) \) that satisfies all of the following properties.

- the domain of \( f \) is all real numbers
- \( f \) is left-continuous but not continuous at \( x = -5 \)
- \( \lim_{x \to -2^-} f(x) = +\infty \) and \( f \) is right-continuous at \( x = -2 \)
- \( f \) is continuous but not differentiable at \( x = 1 \)
- \( f \) is continuous for \( x > 1 \), \( f(3) = 2 \), \( f(5) = -2 \), and \( \lim_{x \to \infty} f(x) = 0 \)

Solution
There are many possible solutions, but the following is perhaps the simplest.

6. For each part, calculate \( f'(x) \).

After calculating the derivative, do not simplify your answer.

(a) \( f(x) = \left( x + \sqrt{3x + 5} \right)^{2/3} \)

(b) \( f(x) = \frac{xe^{-x}}{e^{3x} + \sin(x)} \)
Solution
(a) Use power rule and chain rule (twice!).

\[ f'(x) = \frac{(x \cdot e^{-x} \cdot (-1) + e^{-x})(e^{3x} + \sin(x)) - (xe^{-x})(3e^{3x} + \cos(x))}{(e^{3x} + \sin(x))^2} \]

(b) Use quotient rule and product rule.

\[ g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} \]

The parts of this question are independent of each other.

(a) Given the function \( g(x) \), state the definition of \( g'(x) \).

(b) Let \( f(x) = \sqrt{6x + 1} \). Calculate \( f'(1) \) directly from the definition. Show all work.

\textit{If you simply quote a rule, you will receive no credit. You must use the definition of derivative.}

Solution
(a) \( g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} \)

(b) Start with the definition of derivative, then simplify and cancel.

\[
\begin{align*}
f'(1) &= \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{6(1 + h) + 1} - \sqrt{7}}{h} \\
&= \lim_{h \to 0} \frac{6h}{h(\sqrt{6h + 7} + \sqrt{7})} \\
&= \lim_{h \to 0} \frac{6}{\sqrt{6h + 7} + \sqrt{7}} = \frac{6}{\sqrt{7} + \sqrt{7}} = \frac{3}{\sqrt{7}}
\end{align*}
\]

8. For each limit, calculate the value or show that it does not exist. Show all work.

(a) \( \lim_{x \to 7} \left( \frac{\frac{1}{7} - \frac{1}{x}}{x - 7} \right) \)

(b) \( \lim_{x \to 0} \left( \frac{\sin(7x)}{\tan(2x)} \right) \)

(c) \( \lim_{x \to -1} \left( \frac{|x + 1|}{x + 1} \right) \)
Solution

(a) We have the following work.

\[
\lim_{x \to 7} \left( \frac{1}{x} - \frac{1}{7} \cdot \frac{7x}{x-7} \right) = \lim_{x \to 7} \left( \frac{x - 7}{7x(x-7)} \right) = \lim_{x \to 7} \left( \frac{1}{7x} \right) = \frac{1}{49}
\]

(b) We have the following work.

\[
\lim_{x \to 0} \left( \frac{\sin(7x)}{7x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{7x}{2x} \right) = \left( \lim_{x \to 0} \frac{\sin(7x)}{7x} \right) \cdot \left( \lim_{x \to 0} \frac{2x}{\sin(2x)} \right) \cdot \left( \lim_{x \to 0} \cos(2x) \cdot \frac{7}{2} \right) = 1 \cdot 1 \cdot 1 \cdot \frac{7}{2} = \frac{7}{2}
\]

(c) We have the following work.

\[
\lim_{x \to -1^-} \left( \frac{|x+1|}{x+1} \right) = \lim_{x \to -1^-} \left( \frac{-(x+1)}{x+1} \right) = -1
\]

\[
\lim_{x \to -1^+} \left( \frac{|x+1|}{x+1} \right) = \lim_{x \to -1^+} \left( \frac{+(x+1)}{x+1} \right) = +1
\]

The one-sided limits are not equal, thus the desired limit does not exist.