1. Determine the radius of convergence and interval of convergence for the following power series. Show all work.
\[ \sum_{n=1}^{\infty} \frac{(2x - 3)^n}{n9^n} \]

**Solution**

We first use ratio test to determine the radius of convergence.

\[
\rho = \lim_{n \to \infty} \left| \frac{(2x - 3)^{n+1} \cdot n9^n}{(n+1)9^{n+1} \cdot (2x - 3)^n} \right| = \lim_{n \to \infty} \left| \frac{(2x - 3)^n \cdot n+1 \cdot \frac{1}{9}}{2x - 3} \right|
\]

\[
= |2x - 3| \cdot \frac{1}{9} = \frac{1}{9} |2x - 3|
\]

The series thus converges absolutely for all \( x \) such that \( \rho < 1 \), or for all \( x \) such that

\[
| x - \frac{3}{2} | < \frac{9}{2}
\]

It follows that the radius of convergence is \( R = \frac{9}{2} \). The series is guaranteed to converge absolutely for all \( x \) in the interval \((-3, 6)\). We now check the endpoints individually.

For \( x = -3 \), the series is

\[
\sum_{n=1}^{\infty} \frac{(-6 + 3)^n}{n9^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}
\]

The sequence \( \{\frac{1}{n}\} \) is positive, decreasing, and has limit 0. Thus this series converges by the alternating series test.

For \( x = 6 \), the series is

\[
\sum_{n=1}^{\infty} \frac{(12 - 3)^n}{n9^n} = \sum_{n=1}^{\infty} \frac{1}{n}
\]

This series diverges by the \( p \)-test with \( p = 1 \leq 1 \).

In summary, the given power series has interval of convergence \([-3, 6)\).

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2. Find the first 3 nonzero terms of the MacLaurin series for \( f(x) = \ln(3 + 2x) \). You may use any valid method but you must show all work.

**Solution**

Method 1:

Since we want the first 3 nonzero terms of the series, we need to calculate at least
the first 2 derivatives of \( f \).

\[
\begin{align*}
  f(x) &= \ln(3 + 2x) \\
  f'(x) &= \frac{2}{3 + 2x} \\
  f''(x) &= -\frac{4}{(3 + 2x)^2}
\end{align*}
\]

Hence the first 3 coefficients of the MacLaurin series are

\[
\begin{align*}
  a_0 &= f(0) = \ln(3) \\
  a_1 &= \frac{f'(0)}{1!} = \frac{2}{3} \\
  a_2 &= \frac{f''(0)}{2!} = -\frac{2}{9}
\end{align*}
\]

The first 3 nonzero terms of the Maclaurin series are thus

\[\ln(3) + \frac{2}{3} x - \frac{2}{9} x^2 + \cdots\]

**Method 2:**

Start with the geometric series

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots
\]

Antidifferentiate once to find that

\[-\ln(1-x) = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = C + x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \cdots\]

Substitution of \( x = 0 \) on each side gives \( C = 0 \). Hence we have that

\[\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x - \frac{1}{2} x^2 - \frac{1}{3} x^3 + \cdots\]

Now we observe the following.

\[f(x) = \ln(3 + 2x) = \ln \left( 3 \left( 1 + \frac{2}{3} x \right) \right) = \ln(3) + \ln \left( 1 + \frac{2}{3} x \right)\]
Substituting $x \to -\frac{2}{3}x$ in our MacLaurin series for $\ln(1 - x)$ now gives the desired power series.

\[
\ln(3 + 2x) = \ln(3) - \sum_{n=0}^{\infty} \frac{(-\frac{2}{3}x)^{n+1}}{n+1} \\
= \ln(3) - \left[ \frac{(-\frac{2}{3}x)^1}{1} + \frac{(-\frac{2}{3}x)^2}{2} + \cdots \right] \\
= \ln(3) + \frac{2}{3}x - \frac{2}{9}x^2 + \cdots
\]