1. The total cost of producing $x$ widgets is

$$C(x) = x^3 + 9x^2 + 18x + 200$$

and the selling price per unit is

$$p(x) = 45 - 2x^2$$

What is the optimal price? (That is, what price maximizes total profit?)

**Solution**

The total revenue is $R(x) = xp(x) = 45x - 2x^3$. Thus the marginal cost and marginal revenue are

$$MC(x) = 3x^2 + 18x + 18$$
$$MR(x) = 45 - 6x^2$$

Profit is maximized when $MC = MR$.

$$3x^2 + 18x + 18 = 45 - 6x^2$$
$$9x^2 + 18x - 27 = 0$$
$$9(x + 3)(x - 1) = 0$$

The only solution is $x = 1$ (production cannot be negative). Thus the optimal price is $p(1) = 43$. (Verification that $x = 1$ gives maximum profit is not required for cost-revenue problems.)

2. Suppose the total cost of producing $x$ units is

$$C(x) = 2x^4 - 10x^3 - 18x^2 + x + 5$$

Find the smallest and largest values of marginal cost for $0 \leq x \leq 5$.

**Solution**

The marginal cost is

$$MC(x) = 8x^3 - 30x^2 - 36x + 1$$

To find the local extrema of $MC(x)$, we find the critical numbers of $MC(x)$. Since $MC(x)$ is a polynomial (and hence differentiable for all $x$), the only critical numbers are solutions to the equation $MC'(x) = 0$.

$$0 = MC'(x) = 24x^2 - 60x - 36 = 12(2x + 1)(x - 3) \iff x = 3$$

The minimum and maximum of $MC'(x)$ on the interval $[0, 5]$ must occur at either $x = 3$ or the endpoints.

$$MC(x) = 8x^3 - 30x^2 - 36x + 1 = (4x - 15)(2x^2 - 9) - 134$$

$$MC(0) = 1$$
$$MC(3) = -161$$
$$MC(5) = 71$$
Hence the minimum marginal cost is $-161$ and the maximum marginal cost is $71$.

3. Suppose the total cost of manufacturing $x$ widgets is

$$C(x) = 3x^2 + 5x + 75$$

What level of production minimizes the average cost per unit?

**Solution**

The average cost per unit is

$$AC(x) = \frac{C(x)}{x} = 3x + 5 + \frac{75}{x}$$

To minimize the average cost on the interval $(0, \infty)$, we find the critical numbers. Since $AC(x)$ is differentiable on $(0, \infty)$, the critical numbers are solutions to $AC'(x) = 0$.

$$0 = AC'(x) = 3 - \frac{75}{x^2} \implies x = 5$$

Now observe that $AC''(x) = \frac{150}{x^3} > 0$ for all $x > 0$. Hence the graph of $AC(x)$ is concave up on $(0, \infty)$, whence $x = 5$ gives the global minimum of $AC(x)$.

4. The value of a piece of land $t$ years from now (in the dollars of that year) is

$$Q(t) = Q_0 t^{0.15} e^{0.2\sqrt{t}}$$

where $Q_0 = 100,000$. The prevailing effective annual interest rate is 5%, compounded continuously. How many years from now is the optimal time to sell the land? (That is, how many years from now is the present value of the land a maximum?)

**Solution**

The function $Q(t)$ gives the value of the land $t$ years from now in that year. Hence the value of the land $t$ years from now in today’s dollars is $P(t) = Q(t) e^{-rt}$ where $r = 0.05$. Hence we have

$$P(t) = Q_0 t^{0.15} e^{0.2\sqrt{t}-0.05t}$$

Since $P(t)$ is differentiable on $(0, \infty)$, the critical numbers of $P(t)$ are solutions to $P'(t) = 0$.

$$P'(t) = Q_0 t^{0.15} e^{0.2\sqrt{t}-0.05t} \left(0.1t^{-1/2} - 0.05\right) + Q_0 (0.15) t^{-0.85} e^{0.2\sqrt{t}-0.05t}$$

$$= Q_0 t^{-0.85} e^{0.2\sqrt{t}-0.05t} \left(t(0.1t^{-1/2} - 0.05) + 0.15\right)$$

$$= -0.05 Q_0 t^{-0.85} e^{0.2\sqrt{t}-0.05t} \left(2t^{1/2} - 0.05t + 1.5\right)$$

$$= -0.05 Q_0 t^{-0.85} e^{0.2\sqrt{t}-0.05t} \left(\sqrt{t} + 1\right) \left(\sqrt{t} - 3\right)$$

Hence the only solution to $P'(t) = 0$ is $t = 9$. Now we use first derivative test to verify that $t = 9$ really does give the maximum value. Note that the first few factors of $P'(t)$ are sign-definite,
regardless of the value of \( t \). That is, 
\[
0.05Q_0 \varepsilon^{-0.85} \varepsilon^{0.2\sqrt{t-0.05t}} \delta \delta \left( \sqrt{t} + 1 \right) = \nabla
\]

So determining the sign of \( P'(t) \) reduces to finding the sign of \( (\sqrt{t} - 3) \) only. Thus we have
\[
P'(1) = \bigoplus \cdot (1 - 3) = \bigoplus \bigoplus = \bigoplus
\]
\[
P'(16) = \bigoplus \cdot (4 - 3) = \bigoplus = \bigoplus
\]

This means \( P(t) \) is increasing on \([0, 9)\) and decreasing on \((9, \infty)\). Hence \( t = 9 \) gives the maximum value of \( P(t) \).

5. A tour agency is booking a tour and has 100 people signed up. The price of a ticket is $2000 per person. The agency has booked a plane seating 150 people at a cost of $125,000. Additional costs to the agency are incidental fees of $500 per person. For each $10 that the price is lowered, a new person will sign up. How much should the price be lowered for all participants to maximize the profit to the tour agency?

Solution
Let \( x \) be the number of people signed up and let \( p \) be the price of a ticket. Then \( p \) is a linear function of \( x \) (note the phrase “for each” in the problem). We know that \( p = 2000 \) if \( x = 100 \) and that \( \Delta x = 1 \) if \( \Delta p = -10 \). This means if we write \( p(x) = p_0 + m(x - x_0) \), we have the point \((x_0, p_0) = (100, 2000)\) and the slope \( m = -10 \). Hence
\[
p(x) = 2000 - 10(x - 100) = 3000 - 10x
\]
The total revenue and total cost for the agency are thus
\[
R(x) = xp(x) = 3000x - 10x^2 \\
C(x) = 125000 + 500x
\]
The total profit is maximized when marginal cost is equal to marginal revenue.
\[
MR = MC \implies 3000 - 20x = 500 \implies x = 125
\]
Note that we are maximizing the profit on the interval \( x \in [0, 150] \) since the plane holds at most 150 people. Since \( x = 125 \) is in the valid interval, \( x = 125 \) gives the maximum profit. (Cost-revenue problems do not require verification as long as the candidate level of production is in the valid interval.) The optimal price is thus \( p(125) = 3000 - 1250 = 1750 \). So the price should be lowered by $250.