1. The sum of two numbers is 80. Find the largest possible product.

Solution
We want to maximize \( P(x, y) = xy \) subject to the constraint \( x + y = 80 \). Hence \( y = 80 - x \), and we have to maximize the function

\[
P(x) = x(80 - x) = 80x - x^2
\]

on the interval \((-\infty, \infty)\). Since \( P \) is differentiable everywhere, the only critical numbers are solutions to \( P'(x) = 0 \).

\[
P'(x) = 80 - 2x = 0 \implies x = 40
\]

Now observe that \( P''(x) = -2 \), which is negative for all \( x \). This means the graph of \( P(x) \) is concave down on the interval \((-\infty, \infty)\). Hence \( x = 40 \) gives the global maximum of \( P \). The maximum product is \( P(40) = 1600 \).

2. The sum of two numbers is 10. Find the smallest possible value for the sum of their squares.

Solution
We want to minimize \( S(x, y) = x^2 + y^2 \) subject to the constraint \( x + y = 10 \). Hence \( y = 10 - x \), and we have to minimize the function

\[
S(x) = x^2 + (10 - x)^2
\]

on the interval \((-\infty, \infty)\). Since \( S \) is differentiable everywhere, the only critical numbers are solutions to \( S'(x) = 0 \).

\[
S'(x) = 2x - 2(10 - x) = 0 \implies 4x - 20 = 0 \implies x = 5
\]

Now observe that \( S''(x) = 4 \), which is positive for all \( x \). This means the graph of \( S(x) \) is concave up on the interval \((-\infty, \infty)\). Hence \( x = 5 \) gives the global minimum of \( S \). The minimum sum of squares is \( S(5) = 50 \).

3. Find the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 4, assuming that one side of the rectangle lies on the diameter of the semicircle.

Solution
Let \( x \) be the half-length of the rectangle and let \( y \) be the height. Then we want to maximize the function \( A(x, y) = 2xy \). See the figure below.
By Pythagorean theorem, $x^2 + y^2 = r^2$ (with $r = 4$), whence $y = \sqrt{16 - x^2}$. So we want to maximize the function

$$A(x) = 2x\sqrt{16 - x^2}$$

on the interval $[0, 4]$. Since $A$ is differentiable on $(0, 4)$, the only critical numbers are the endpoints $x = 0$ and $x = 4$, and solutions to $A'(x) = 0$.

$$A'(x) = 2x \cdot \frac{-2x}{2\sqrt{16 - x^2}} + 2\sqrt{16 - x^2} = \frac{32 - 4x^2}{\sqrt{16 - x^2}} = 0 \implies x = \sqrt{8}$$

We now use the closed bounded interval test to verify $x = \sqrt{8}$ gives the maximum. Observe that $A(0) = A(4) = 0$ and $A(\sqrt{8})$ is clearly a positive number. Hence the maximum of $A$ occurs at $x = \sqrt{8}$.

The dimensions of the rectangle of maximum area are $2\sqrt{8}$ (length) by $\sqrt{8}$ (height).

4. Find the dimensions of the rectangle of largest area whose lower vertices lie on the $x$-axis and whose upper vertices lie on the graph of $y = e^{-x^2}$.

**Solution**

Let $x$ be the half-length of the rectangle and let $y$ be the height. (This means the upper left vertex has coordinates $(-x, y)$ and the upper right vertex has coordinates $(x, y)$.) Then we want to maximize the function $A(x, y) = 2xy$. Since the upper vertices of the rectangle lie on the given graph, we must have $y = e^{-x^2}$. Hence we want to maximize the function

$$A(x) = 2xe^{-x^2}$$

on the interval $[0, \infty)$. Since $A$ is differentiable everywhere, the only critical numbers are the endpoint $x = 0$ and solutions to $A'(x) = 0$.

$$A'(x) = 2xe^{-x^2}(-2x) + 2e^{-x^2} = 2e^{-x^2}(1 - 2x^2) = 0 \implies x = \frac{1}{\sqrt{2}}$$

We will use first derivative test to verify we have found the $x$-value that gives the maximum. Note that $A'(0) = 2 > 0$ and $A'(1) = \frac{-2}{e} < 0$. Hence $A(x)$ is increasing on $[0, \frac{1}{\sqrt{2}})$ and decreasing on $(\frac{1}{\sqrt{2}}, \infty)$. Hence $A(x)$ has a global maximum at $x = \frac{1}{\sqrt{2}}$ on the interval $[0, \infty)$. The dimensions of the rectangle of maximum area are $\sqrt{2}$ (length) by $\frac{1}{\sqrt{2}}$ (height).

5. A farmer is constructing a rectangular fence on a straight river. The side of the rectangle bordering the river does not need any fencing. If the farmer has 1000 feet of fencing, what is the largest possible area he may enclose?

**Solution**

Let $x$ be the length of the plot perpendicular to the river and let $y$ be the length parallel to the river. We want to maximize the function $A(x, y) = xy$. See the figure below.
We must maximize the area subject to the constraint $2x + y = 1000$, whence $y = 1000 - 2x$. So we want to maximize the function

$$A(x) = x(1000 - 2x) = 1000x - 2x^2$$

on the interval $[0, \infty)$. Since $A$ is differentiable on $(0, \infty)$, the only critical numbers are the endpoint $x = 0$ and solutions to $A'(x) = 0$.

$$A'(x) = 1000 - 4x = 0 \implies x = 250$$

Now observe that $A''(x) = -2$, which is negative for all $x$. Hence $x = 500$ gives a global maximum of $A(x)$ on the interval $[0, \infty)$.

The maximum possible area is $A(500) = 125,000 \text{ ft}^2$.

6. A farmer with 1600 feet of fencing wants to enclose a rectangular area and then divide it into four equal-area pens with fencing parallel to one side of the rectangle. What is the largest possible area that a single pen can enclose?

**Solution**

Let $x$ be the length of each pen (so the length of the entire enclosure is $4x$) and let $y$ be the width of each pen (the widths are parallel to each other, so the width of the entire enclosure is also $y$). We want to maximize the function $A(x, y) = xy$. See the figure below.

We must maximize the area subject to the constraint $8x + 5y = 1600$, whence $y = \frac{1}{5}(1600 - 8x)$. So we want to maximize the function

$$A(x) = x \cdot \frac{1}{5}(1600 - 8x) = \frac{1}{5}(1600x - 8x^2)$$
on the interval $[0, 200]$. (These endpoints correspond, respectively, to the degenerate cases $x = 0$ and $y = 0$.) Since $A$ is differentiable on $(0, 200)$ the only critical numbers are the endpoints $x = 0$ and $x = 200$ and solutions to $A'(x) = 0$.

$$A'(x) = \frac{1}{5}(1600 - 16x) = 0 \implies x = 100$$

We now use the closed bounded interval test to verify $x = 100$ gives the maximum. Observe that $A(0) = A(200) = 0$ and $A(100)$ is clearly a positive number. Hence the maximum of $A$ occurs at $x = 100$.

The maximum area of a single pen is $A(100) = 16,000 \text{ ft}^2$.

7. A truck is 250 miles east of a sports car and is traveling west at a constant speed of 60 miles per hour. Meanwhile, the sports car is going north at 80 miles per hour. When will the truck and car be closest to each other? What is the minimum distance between them?

**Solution**

We consider a coordinate system in which the sports car is initially at the origin. The truck travels along the $x$-axis, whence the coordinates of its position are $(x(t), 0)$. We know that $x(0) = 250$ and $\frac{dx}{dt} = -60$ for all $t$. Hence $x(t) = 250 - 60t$. The car travels along the $y$-axis, whence the coordinates of its position are $(0, y(t))$. We know that $y(0) = 0$ and $\frac{dy}{dt} = 80$ for all $t$. Hence $y(t) = 80t$. In summary, the coordinates of each vehicle at time $t$ are given by

\[
\begin{align*}
\text{truck} : & \quad T = (250 - 60t, 0) \\
\text{car} : & \quad C = (0, 80t)
\end{align*}
\]

The distance $D$ between the car and truck at time $t$ is thus

$$D(t) = \sqrt{(250 - 60t)^2 + (80t)^2}$$

We want to minimize $D(t)$ on the interval $[0, \infty)$. Since $D$ is differentiable everywhere, the only critical numbers of $D$ are the endpoint $t = 0$ and solutions to $D'(t) = 0$.

$$D'(t) = \frac{2(250 - 60t)(-60) + 2(80t)(80)}{2\sqrt{(250 - 60t)^2 + (80t)^2}} = \frac{10,000t - 15,000}{\sqrt{(250 - 60t)^2 + (80t)^2}} = 0 \implies t = 1.5$$

We will use first derivative test to verify we have found the $t$-value that gives the minimum. Note that $D'(0) = \frac{10,000}{15,000} < 0$ and $D'(2) = \frac{10,000}{15,000} > 0$. Hence $D(t)$ is decreasing on $[0, 1.5)$ and increasing on $(1.5, \infty)$. Hence $D(t)$ has a global minimum at $t = 1.5$ on the interval $[0, \infty)$.

The truck and car are closest to each other 1.5 hours later, and their minimum separation is $D(1.5) = 200$ miles.

8. Suppose we want to construct a rectangular aquarium that must hold a volume of 4000 in$^3$. The length of the base will be twice the width of the base. The top and bottom bases of the tank cost $1.50/\text{in}^2$. Each of the sides of the tank costs $3/\text{in}^2$. Find the dimensions (length, width, height) of the cheapest tank.
Solution
Let $\ell$, $w$, and $h$ denote the length, width, and height of the aquarium. The cost of the top and bottom bases is $1.5(2\ell w) = 3\ell w$. The cost of the sides is $3(2\ell h + 2wh) = 6h(\ell + w)$. So we want to minimize the total cost function

$$C(\ell, w, h) = 3\ell w + 6h(\ell + w)$$

One constraint is that $\ell = 2w$, and the second constraint is that $\ell wh = 4000$. Substituting $\ell = 2w$ into the volume constraint and solving for $h$ gives $h = \frac{2000}{w^2}$. Now writing $\ell$ and $h$ in terms of $w$ in the cost function shows that we have to minimize the function

$$C(w) = 6w^2 + \frac{36,000}{w}$$

on the interval $(0, \infty)$. Since $C$ is differentiable on $(0, \infty)$, the only critical numbers are solutions to $C'(w) = 0$.

$$C'(w) = 12w - \frac{36,000}{w^2} = 0 \implies w = \sqrt[3]{3000} = 10\sqrt[3]{3}$$

Now observe that $C''(w) = 12 + \frac{72,000}{w^3}$, which is positive for all $w$ in $(0, \infty)$. This means the graph of $S(w)$ is concave up on the interval $(0, \infty)$. Hence $w = 10\sqrt[3]{3}$ gives the global minimum of $C$.

The dimensions of the cheapest tank are $\ell = 20\sqrt[3]{3}$ in. (length), $w = 10\sqrt[3]{3}$ in. (width), and $h = \frac{2000}{w^2}$ in. (height).