1. For each part, determine whether the hypotheses of the mean value theorem (MVT) are satisfied for the given function and interval. Explain your answer. If the hypotheses are satisfied, find all values of $c$ guaranteed to exist by the MVT.

(a) $f(x) = x^3 + x$ on $[1, 2]$

(b) $f(x) = (x^2 - 4)^{1/3}(x - 3)$ on $[-1, 4]$

(c) $f(x) = \ln(x)$ on $[\frac{1}{2}, 2]$

Solution

(a) The function $f$ is continuous and differentiable on all intervals, so the hypotheses of the MVT are satisfied on any interval. Hence we must find all values of $c$ in the interval $(0, 2)$ that satisfy the equation

$$\frac{f(2) - f(1)}{2 - 1} = f'(c)$$

$$8 = 3c^2 + 1$$

Solving this quadratic equation gives $c = \pm \sqrt{\frac{7}{3}}$. The only value of $c$ guaranteed to exist by the MVT is $c = \sqrt{\frac{7}{3}}$ (the negative root does not lie in the interval $(1, 2)$).

(b) The function $f$ is not differentiable at $x = -2$ or $x = 2$. Since 2 is contained in the interval $(-1, 4)$, the hypotheses of the MVT are not satisfied by $f$ on this interval.

(c) The function $f$ is continuous and differentiable on its domain (positive numbers), so the hypotheses of the MVT are satisfied on the interval $[\frac{1}{2}, 2]$. Hence we must find all values of $c$ in $(\frac{1}{2}, 2)$ that satisfy the equation

$$\frac{f(2) - f(\frac{1}{2})}{2 - \frac{1}{2}} = f'(c)$$

$$\frac{\ln(2) - \ln(\frac{1}{2})}{\frac{3}{2}} = \frac{1}{c}$$

$$\frac{2\ln(4)}{3} = \frac{1}{c}$$

The value of $c$ guaranteed to exist by the MVT is $c = \frac{3}{2\ln(4)}$. 