1. For each part, find the absolute maximum and the absolute minimum of the function $f$ on the given interval.

(a) $f(x) = x^4 - 8x^2$ on $[-3, 3]$
(b) $f(x) = x^3 + 3x^2 - 24x - 72$ on $[-4, 4]$
(c) $f(x) = \sqrt{x(x-5)}^{1/3}$ on $[0, 6]$
(d) $f(x) = e^{-x}\sin(x)$ on $[0, 2\pi]$
(e) $f(x) = x(\ln(x) - 5)^2$ on $[e^{-4}, e^4]$
(f) $f(x) = \begin{cases} 9 - 4x & \text{if } x < 1 \\ -x^2 + 6x & \text{if } x \geq 1 \end{cases}$ on $[0, 4]$

Solution

(a) The function $f$ is differentiable everywhere. So we solve $f'(x) = 0$.

$$0 = f'(x) = 4x^3 - 16x$$
$$0 = 4x(x-2)(x+2)$$

Hence the critical points are $x = -2, x = 0, \text{ and } x = 2$. Checking the critical values and the endpoint values gives the following.

$$f(x) = x^4 - 8x^2 = x^2(x^2 - 8)$$
$$f(-3) = 9$$
$$f(-2) = -16$$
$$f(0) = 0$$
$$f(2) = -16$$
$$f(3) = 9$$

The maximum value of $f$ on $[-3, 3]$ is 9 and the minimum value is $-16$.

(b) The function $f$ is differentiable everywhere. So we solve $f'(x) = 0$.

$$0 = f'(x) = 3x^2 + 6x - 24$$
$$0 = 3(x-2)(x+4)$$

Hence the critical points are $x = -4$ and $x = 2$. Checking the critical values and the endpoint values gives the following.

$$f(x) = x^3 + 3x^2 - 24x - 72 = (x^2 - 24)(x+3)$$
$$f(-4) = (-8)(-1) = 8$$
$$f(2) = (-20)(5) = -100$$
$$f(4) = (-8)(7) = -56$$

The maximum value of $f$ on $[-4, 4]$ is 8 and the minimum value is $-100$.

(c) The function $f$ is not differentiable at $x = 5$, hence $x = 5$ is a critical point. To find the
other critical points we solve the equation \( f'(x) = 0 \).

\[
0 = f'(x) = x^{1/2} \cdot \frac{1}{3} (x - 5)^{-2/3} + \frac{1}{2} x^{-1/2} (x - 5)^{1/3}
\]

\[
0 = \frac{1}{6} x^{-1/2} (x - 5)^{-2/3} (2x + 3(x - 5))
\]

\[
0 = \frac{1}{6} x^{-1/2} (x - 5)^{-2/3} (5x - 15)
\]

Solving this equation thus gives \( 5x - 15 = 0 \) (that is, \( x = 3 \)). Checking the critical values and the endpoint values gives the following.

\[
f(x) = x^{1/2} (x - 5)^{1/3}
\]

\[
f(0) = 0
\]

\[
f(3) = 3^{1/2} (-2)^{1/3} \quad \text{(negative number)}
\]

\[
f(5) = 0
\]

\[
f(6) = 6^{1/2} \quad \text{(positive number)}
\]

The maximum value of \( f \) on \([0, 6]\) is \( 6^{1/2} \) and the minimum value is \( 3^{1/2} (-2)^{1/3} \).

(d) The function \( f \) is differentiable everywhere. So we solve \( f'(x) = 0 \).

\[
0 = f'(x) = e^{-x} \cos(x) - e^{-x} \sin(x)
\]

\[
0 = e^{-x} (\cos(x) - \sin(x))
\]

Solving this equation thus gives \( \cos(x) - \sin(x) = 0 \) (that is, \( \tan(x) = 1 \)). In the interval \([0, 2\pi]\) the equation \( \tan(x) = 1 \) has solutions \( x = \frac{\pi}{4} \) and \( \frac{5\pi}{4} \). Checking the critical values and the endpoint values gives the following.

\[
f(x) = e^{-x} \sin(x)
\]

\[
f(0) = 0
\]

\[
f \left( \frac{\pi}{4} \right) = e^{-\pi/4} \cdot \frac{1}{\sqrt{2}} \quad \text{(positive number)}
\]

\[
f \left( \frac{5\pi}{4} \right) = -e^{-5\pi/4} \cdot \frac{1}{\sqrt{2}} \quad \text{(negative number)}
\]

\[
f(2\pi) = 0
\]

The maximum value of \( f \) on \([0, 2\pi]\) is \( \frac{e^{-\pi/4}}{\sqrt{2}} \) and the minimum value is \( -\frac{e^{-5\pi/4}}{\sqrt{2}} \).

(e) The function \( f \) is differentiable on its domain. So we solve \( f'(x) = 0 \).

\[
0 = f'(x) = x \cdot 2 \ln(x) - 5 \cdot \frac{1}{x} + (\ln(x) - 5)^2
\]

\[
0 = 2 (\ln(x) - 5) + (\ln(x) - 5)^2
\]

\[
0 = (\ln(x) - 5) (2 + \ln(x) - 5)
\]

\[
0 = (\ln(x) - 5) (\ln(x) - 3)
\]
Solving this equation thus gives \( \ln(x) - 5 = 0 \) (that is, \( x = e^5 \)) or \( \ln(x) - 3 = 0 \) (that is, \( x = e^3 \)). The only critical point is thus \( x = e^3 \) (\( e^5 \) is not in the interval \([e^{-4}, e^4]\)). Checking the critical values and the endpoint values gives the following.

\[
\begin{align*}
f(x) &= x(\ln(x) - 5)^2 \\
f(e^{-4}) &= e^{-4}(-4 - 5)^2 = \frac{81}{e^4} \\
f(e^3) &= e^3(3 - 5)^2 = 4e^3 \\
f(e^4) &= e^4(4 - 5)^2 = e^4
\end{align*}
\]

To determine which value is the largest and which is the smallest, we look at the ratios of the above values. We will use the fact that \( 2 < e < 4 \).

\[
\frac{f(e^3)}{f(e^4)} = \frac{4e^3}{e^4} = \frac{4}{e} > e = 1
\]

Hence \( f(e^3) > f(e^4) \). We also have

\[
\frac{f(e^4)}{f(e^{-4})} = \frac{e^4}{\frac{81}{e^4}} = \frac{e^8}{81} = \frac{2^8}{81} = \frac{256}{81} > 1
\]

Hence \( f(e^4) > f(e^{-4}) \). Putting this all together we find the following.

\[
4e^3 > e^4 > \frac{81}{e^4}
\]

The maximum value of \( f \) on \([e^{-4}, e^4]\) is \( 4e^3 \) and the minimum value is \( \frac{81}{e^4} \).

(f) First observe that \( f \) is continuous (the left-limit, right-limit, and function value are all equal to 5 at \( x = 1 \), the only suspicious point). So the extreme value theorem does apply to \( f \) on the interval \([0, 4]\).

The derivative of \( f \) is given by

\[
f'(x) = \begin{cases} 
-4, & x < 1 \\
-2x + 6, & x > 1 
\end{cases}
\]

The function \( f \) is not differentiable at \( x = 1 \). We may verify this by computing the following limit.

\[
f'(1) = \lim_{h \to 0} \left( \frac{f(1 + h) - f(1)}{h} \right) = \lim_{h \to 0} \left( \frac{f(1 + h) - 5}{h} \right)
\]

Since \( f(1 + h) \) is defined differently depending on whether \( h \) is negative or positive, we
compute the one-sided limits.

\[
\lim_{h \to 0^-} \left( \frac{f(1+h) - 5}{h} \right) = \lim_{h \to 0^-} \left( \frac{9 - 4(1+h) - 5}{h} \right) = \lim_{h \to 0^-} \left( \frac{-4h}{h} \right) = \lim_{h \to 0^-} (-4) = -4
\]

\[
\lim_{h \to 0^+} \left( \frac{f(1+h) - 5}{h} \right) = \lim_{h \to 0^+} \left( \frac{-(1+h)^2 + 6(1+h) - 5}{h} \right) = \lim_{h \to 0^+} \left( \frac{-h^2 + 4h}{h} \right) = \lim_{h \to 0^+} (-h + 4) = 4
\]

Since the two one-sided limits are not equal, \(f'(1)\) does not exist. This means \(x = 1\) is a critical point of \(f\) on the interval \([0, 4]\).

To find any other critical point of \(f\) we solve the equation \(f'(x) = 0\). Note that the “first piece” of \(f'(x)\) (i.e., \(-4\)) is never equal to 0. Hence we only set the “second piece” of \(f'(x)\) (i.e., \(-2x + 6\)) equal to 0. The equation \(-2x + 6 = 0\) has the solution \(x = 3\). (Also observe that \(x = 3\) lies in the interval \(x > 1\), i.e., the valid \(x\)-values for the “second piece” of \(f'(x)\).)

Checking the critical values and endpoint values gives the following.

\[
f(x) = \begin{cases} 
9 - 4x, & x < 1 \\
-x^2 + 6x, & x \geq 1 
\end{cases}
\]

\(f(0) = 9\)
\(f(1) = 5\)
\(f(3) = 9\)
\(f(4) = 8\)

The maximum value of \(f\) on \([0, 4]\) is 9 and the minimum value is 5.

2. A particle moves along the \(x\) axis with position

\[
x(t) = t^4 - 2t^3 - 12t^2 + 60t - 10
\]

Find the particle’s minimum velocity for \(0 \leq t \leq 3\).

**Solution**

The velocity of the particle is

\[
v(t) = \frac{dx}{dt} = 4t^3 - 6t^2 - 24t + 60
\]

We must find the maximum value of \(v(t)\). Since \(v(t)\) is differentiable on all intervals, the critical points of \(v(t)\) are those values of \(t\) for which \(v'(t) = 0\).

\[
0 = v'(t) = 12t^2 - 12t - 24
\]

\[
0 = 12(t^2 - t - 2) = 12(t - 2)(t + 1)
\]
The only critical point is $t = 2$ (the value $t = -1$ is not in the interval $[0, 3]$). Now we check the values of $v$ at the critical point and the endpoints of the interval.

\[
\begin{align*}
v(0) &= 60 \\
v(2) &= 20 \\
v(3) &= 42
\end{align*}
\]

Hence the particle’s minimum velocity is $v(2) = 20$. 