1. For each part, find the absolute maximum and the absolute minimum of the function $f$ on the given interval.

(a) $f(x) = x^4 - 8x^2$ on $[-3, 3]$  
(b) $f(x) = x^3 + 3x^2 - 24x - 72$ on $[-4, 4]$  
(c) $f(x) = \sqrt{x(x - 5)}^{1/3}$ on $[0, 6]$  
(d) $f(x) = e^{-x} \sin(x)$ on $[0, 2\pi]$  
(e) $f(x) = x(\ln(x) - 5)^2$ on $[e^{-4}, e^4]$  
(f) $f(x) = \begin{cases} 9 - 4x, & x < 1 \\ -x^2 + 6x, & x \geq 1 \end{cases}$ on $[0, 4]$

Solution

(a) The function $f$ is differentiable everywhere. So we solve $f'(x) = 0$. 

$$0 = f'(x) = 4x^3 - 16x$$
$$0 = 4x(x - 2)(x + 2)$$

Hence the critical points are $x = -2, x = 0,$ and $x = 2$. Checking the critical values and the endpoint values gives the following.

$$f(x) = x^4 - 8x^2 = x^2(x^2 - 8)$$
$$f(-3) = 9$$
$$f(-2) = -16$$
$$f(0) = 0$$
$$f(2) = -16$$
$$f(3) = 9$$

The maximum value of $f$ on $[-3, 3]$ is 9 and the minimum value is $-16$.

(b) The function $f$ is differentiable everywhere. So we solve $f'(x) = 0$. 

$$0 = f'(x) = 3x^2 + 6x - 24$$
$$0 = 3(x - 2)(x + 4)$$

Hence the critical points are $x = -4$ and $x = 2$. Checking the critical values and the endpoint values gives the following.

$$f(x) = x^3 + 3x^2 - 24x - 72 = (x^2 - 24)(x + 3)$$
$$f(-4) = (-8)(-1) = 8$$
$$f(2) = (-20)(5) = -100$$
$$f(4) = (-8)(7) = -56$$

The maximum value of $f$ on $[-4, 4]$ is 8 and the minimum value is $-100$.

(c) The function $f$ is not differentiable at $x = 5$, hence $x = 5$ is a critical point. To find the
other critical points we solve the equation $f'(x) = 0$.

$$0 = f'(x) = x^{1/2} \cdot \frac{1}{3}(x - 5)^{-2/3} + \frac{1}{2}x^{-1/2}(x - 5)^{1/3}$$

$$0 = \frac{1}{6}x^{-1/2}(x - 5)^{-2/3} (2x + 3(x - 5))$$

$$0 = \frac{1}{6}x^{-1/2}(x - 5)^{-2/3}(5x - 15)$$

Solving this equation thus gives $5x - 15 = 0$ (that is, $x = 3$). Checking the critical values and the endpoint values gives the following.

$$f(x) = x^{1/2}(x - 5)^{1/3}$$

$$f(0) = 0$$

$$f(3) = 3^{1/2}(-2)^{1/3} \quad \text{(negative number)}$$

$$f(5) = 0$$

$$f(6) = 6^{1/2} \quad \text{(positive number)}$$

The maximum value of $f$ on $[0, 6]$ is $6^{1/2}$ and the minimum value is $3^{1/2}(-2)^{1/3}$.

(d) The function $f$ is differentiable everywhere. So we solve $f'(x) = 0$.

$$0 = f'(x) = e^{-x} \cos(x) - e^{-x} \sin(x)$$

$$0 = e^{-x} (\cos(x) - \sin(x))$$

Solving this equation thus gives $\cos(x) - \sin(x) = 0$ (that is, $\tan(x) = 1$). In the interval $[0, 2\pi]$ the equation $\tan(x) = 1$ has solutions $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$. Checking the critical values and the endpoint values gives the following.

$$f(x) = e^{-x} \sin(x)$$

$$f(0) = 0$$

$$f \left( \frac{\pi}{4} \right) = e^{-\pi/4} \cdot \frac{1}{\sqrt{2}} \quad \text{(positive number)}$$

$$f \left( \frac{5\pi}{4} \right) = -e^{-5\pi/4} \cdot \frac{1}{\sqrt{2}} \quad \text{(negative number)}$$

$$f(2\pi) = 0$$

The maximum value of $f$ on $[0, 2\pi]$ is $\frac{e^{-\pi/4}}{\sqrt{2}}$ and the minimum value is $-\frac{e^{-5\pi/4}}{\sqrt{2}}$.

(e) The function $f$ is differentiable on its domain. So we solve $f'(x) = 0$.

$$0 = f'(x) = x \cdot 2 (\ln(x) - 5) \cdot \frac{1}{x} + (\ln(x) - 5)^2$$

$$0 = 2 (\ln(x) - 5) + (\ln(x) - 5)^2$$

$$0 = (\ln(x) - 5) (2 + \ln(x) - 5)$$

$$0 = (\ln(x) - 5) (\ln(x) - 3)$$
Solving this equation thus gives $\ln(x) - 5 = 0$ (that is, $x = e^5$) or $\ln(x) - 3 = 0$ (that is, $x = e^3$). The only critical point is thus $x = e^3$ ($e^5$ is not in the interval $[e^{-4}, e^4]$).

Checking the critical values and the endpoint values gives the following.

\[
f(x) = x (\ln(x) - 5)^2
\]
\[
f(e^{-4}) = e^{-4}(-4 - 5)^2 = \frac{81}{e^4}
\]
\[
f(e^3) = e^3(3 - 5)^2 = 4e^3
\]
\[
f(e^4) = e^4(4 - 5)^2 = e^4
\]

To determine which value is the largest and which is the smallest, we look at the ratios of the above values. We will use the fact that $2 < e < 4$.

\[
\frac{f(e^3)}{f(e^4)} = \frac{4e^3}{e^4} = \frac{4}{e} > \frac{e}{e} = 1
\]

Hence $f(e^3) > f(e^4)$. We also have

\[
\frac{f(e^4)}{f(e^{-4})} = \frac{e^4}{\frac{81}{e^4}} = \frac{e^8}{81} > \frac{2^8}{81} = \frac{256}{81} > 1
\]

Hence $f(e^4) > f(e^{-4})$. Putting this all together we find the following.

\[
4e^3 > e^4 > \frac{81}{e^4}
\]

The maximum value of $f$ on $[e^{-4}, e^4]$ is $4e^3$ and the minimum value is $\frac{81}{e^4}$.

(f) First observe that $f$ is continuous (the left-limit, right-limit, and function value are all equal to 5 at $x = 1$, the only suspicious point). So the extreme value theorem does apply to $f$ on the interval $[0, 4]$.

The derivative of $f$ is given by

\[
f'(x) = \begin{cases} 
-4, & x < 1 \\
-2x + 6, & x > 1 
\end{cases}
\]

The function $f$ is not differentiable at $x = 1$. We may verify this by computing the following limit.

\[
f'(1) = \lim_{h \to 0} \left( \frac{f(1 + h) - f(1)}{h} \right) = \lim_{h \to 0} \left( \frac{f(1 + h) - 5}{h} \right)
\]

Since $f(1 + h)$ is defined differently depending on whether $h$ is negative or positive, we
compute the one-sided limits.

\[
\lim_{h \to 0^-} \left( \frac{f(1 + h) - 5}{h} \right) = \lim_{h \to 0^-} \left( \frac{9 - 4(1 + h) - 5}{h} \right) = \lim_{h \to 0^-} \left( \frac{-4h}{h} \right) = \lim_{h \to 0^-} (-4) = -4
\]

\[
\lim_{h \to 0^+} \left( \frac{f(1 + h) - 5}{h} \right) = \lim_{h \to 0^+} \left( \frac{-(1 + h)^2 + 6(1 + h) - 5}{h} \right) = \lim_{h \to 0^+} \left( \frac{-h^2 + 4h}{h} \right) = \lim_{h \to 0^+} (-h + 4) = 4
\]

Since the two one-sided limits are not equal, \( f'(1) \) does not exist. This means \( x = 1 \) is a critical point of \( f \) on the interval \([0, 4]\).

To find any other critical point of \( f \) we solve the equation \( f'(x) = 0 \). Note that the “first piece” of \( f'(x) \) (i.e., \(-4\)) is never equal to 0. Hence we only set the “second piece” of \( f'(x) \) (i.e., \(-2x + 6\)) equal to 0. The equation \(-2x + 6 = 0\) has the solution \( x = 3 \). (Also observe that \( x = 3 \) lies in the interval \( x > 1 \), i.e., the valid \( x \)-values for the “second piece” of \( f'(x) \).)

Checking the critical values and endpoint values gives the following.

\[
f(x) = \begin{cases} 
9 - 4x & , \quad x < 1 \\
-x^2 + 6x & , \quad x \geq 1 
\end{cases}
\]

\[
f(0) = 9 \\
f(1) = 5 \\
f(3) = 9 \\
f(4) = 8
\]

The maximum value of \( f \) on \([0, 4]\) is 9 and the minimum value is 5.

2. A particle moves along the \( x \) axis with position

\[
x(t) = t^4 - 2t^3 - 12t^2 + 60t - 10
\]

Find the particle’s minimum velocity for \( 0 \leq t \leq 3 \).

**Solution**

The velocity of the particle is

\[
v(t) = \frac{dx}{dt} = 4t^3 - 6t^2 - 24t + 60
\]

We must find the maximum value of \( v(t) \). Since \( v(t) \) is differentiable on all intervals, the critical points of \( v(t) \) are those values of \( t \) for which \( v'(t) = 0 \).

\[
0 = v'(t) = 12t^2 - 12t - 24 \\
0 = 12(t^2 - t - 2) = 12(t - 2)(t + 1)
\]
The only critical point is $t = 2$ (the value $t = -1$ is not in the interval $[0, 3]$). Now we check the values of $v$ at the critical point and the endpoints of the interval.

\[
\begin{align*}
v(0) &= 60 \\
v(2) &= 20 \\
v(3) &= 42
\end{align*}
\]

Hence the particle’s minimum velocity is $v(2) = 20$. 