1. Use a linear approximation to estimate the value of each of the following. 
You must express your answer as a single exact rational number.

(a) $e^{0.1}$
(b) $\ln(1.04)$
(c) $\frac{1}{\sqrt[3]{25}}$
(d) $(\sec \left( \frac{\pi}{4} - 0.02 \right))^2$

Solution

(a) Let $f(x) = e^x$. Our goal is to estimate $f(0.1)$ using the tangent line to $f(x)$ at $x = 0$.

\[
\begin{align*}
  f(0) &= e^0 = 1 \\
  f'(x) &= e^x \\
  f'(0) &= e^0 = 1
\end{align*}
\]

Hence an equation of the tangent line to $f(x)$ at $x = 0$ is

\[
y = 1 + x
\]

Substituting $x = 0.1$ into the tangent line gives the desired approximation.

\[
e^{0.1} = f(0.1) \approx 1 + 0.1 = 1.1
\]

(b) Let $f(x) = \ln(x)$. Our goal is to estimate $f(1.04)$ using the tangent line to $f(x)$ at $x = 1$.

\[
\begin{align*}
  f(1) &= \ln(1) = 0 \\
  f'(x) &= \frac{1}{x} \\
  f'(1) &= \frac{1}{1} = 1
\end{align*}
\]

Hence an equation of the tangent line to $f(x)$ at $x = 1$ is

\[
y = x - 1
\]

Substituting $x = 1.04$ into the tangent line gives the desired approximation.

\[
\ln(1.04) = f(1.04) \approx 1.04
\]

(c) Let $f(x) = x^{-1/3}$. Our goal is to estimate $f(25)$ using the tangent line to $f(x)$ at $x = 27$.

\[
\begin{align*}
  f(27) &= 27^{-1/3} = \frac{1}{3} \\
  f'(x) &= -\frac{1}{3}x^{-4/3} \\
  f'(27) &= -\frac{1}{3} \cdot 27^{-4/3} = -\frac{1}{3} \cdot \frac{1}{3^4} = -\frac{1}{243}
\end{align*}
\]
Hence an equation of the tangent line to $f(x)$ at $x = 27$ is

$$y = \frac{1}{3} - \frac{1}{243}(x - 27)$$

Substituting $x = 25$ into the tangent line gives the desired approximation.

$$\frac{1}{\sqrt{25}} = f(25) \approx \frac{1}{3} - \frac{1}{243}(-2) = \frac{81}{243} + \frac{2}{243} = \frac{83}{243}$$

(d) Let $f(x) = \sec(x)^2$. Our goal is to estimate $f\left(\frac{\pi}{4} - 0.02\right)$ using the tangent line to $f(x)$ at $x = \frac{\pi}{4}$.

$$f\left(\frac{\pi}{4}\right) = \sec\left(\frac{\pi}{4}\right)^2 = (\sqrt{2})^2 = 2$$
$$f'(x) = 2 \sec(x) \cdot \sec(x) \tan(x) = 2 \sec(x)^2 \tan(x)$$
$$f'\left(\frac{\pi}{4}\right) = 2 \sec\left(\frac{\pi}{4}\right)^2 \tan\left(\frac{\pi}{4}\right) = 2 \cdot 2 \cdot 1 = 4$$

Hence an equation of the tangent line to $f(x)$ at $x = \frac{\pi}{4}$ is

$$y = 2 + 4\left(x - \frac{\pi}{4}\right)$$

Substituting $x = \frac{\pi}{4} - 0.02$ into the tangent line gives the desired approximation.

$$\left(\sec\left(\frac{\pi}{4} - 0.02\right)\right)^2 = f\left(\frac{\pi}{4} - 0.02\right) \approx 2 + 4(-0.02) = 2 - 0.08 = 1.92$$

2. A manufacturer’s total cost (in dollars) when the level of production is $q$ units is

$$C(q) = q^5 - 2q^3 + 3q^2 - 2$$

The current level of production is 3 units, and the manufacturer is planning to increase this to 3.01 units. Estimate how the total cost will change as a result.

**Solution**

The exact change in cost is

$$\Delta C = C(3.01) - C(3)$$

But using a linear approximation, this change can be estimated using the marginal cost.

$$\Delta C = C(3.01) - C(3) \approx C'(3) \cdot (0.01)$$

(Alternatively, $C(3.01)$ is estimated using the tangent line to $C(q)$ at $q = 3$.)

$$C'(q) = 5q^4 - 6q^2 + 6q$$
$$C'(3) = 5 \cdot 3^4 - 6 \cdot 3^2 + 6 \cdot 3 = 405 - 54 + 6 = 369$$

Hence our estimation of the change in cost is

$$\Delta C \approx (369)(0.01) = 3.69$$
The cost will increase by approximately 3.69 dollars.

3. A manufacturer’s total cost (in dollars) when the level of production is $q$ units is

$$C(q) = 3q^2 + q + 500$$

(a) What is the exact cost of manufacturing the 41st unit?

(b) Use marginal analysis to estimate the cost of manufacturing the 41st unit.

Solution
(a) The exact cost is

$$\Delta C = C(41) - C(40) = 3 \cdot (41^2 - 40^2) + (41 - 40)$$
$$= 3(41 - 40)(41 + 40) + 1 = 3(81) + 1 = 244$$

dollars.

(b) The exact cost of producing one more unit is estimated using the marginal cost. That is,

$$\Delta C = C(41) - C(40) \approx C'(40) \cdot 1$$

(Alternatively, $C(41)$ is estimated using the tangent line to $C(q)$ at $q = 40$.)

$$C'(q) = 6q + 1$$
$$C'(40) = 241$$

Hence our estimation of the cost of producing the 41st unit is

$$\Delta C \approx 241$$
dollars.

4. You measure the radius of a sphere to be 6 inches, and then you use your measurement to calculate the volume of the sphere with the formula $V = \frac{4\pi}{3} r^3$. If your measurement of the radius is accurate to within 1%, approximately how accurate (to the nearest percent) is your calculation of the volume?

Solution
Let $r_0$ denote the measured radius and let $V_0$ denote the volume as calculated from the measured radius. (That is, $r_0 = 6$ and $V_0 = \frac{4\pi}{3} r_0^3 = 288\pi$ in this problem.) Let $r$ and $V$ denote the exact radius and volume of the sphere. (So $V = \frac{4\pi}{3} r^3$.)

Recall that the error between the measured volume and the exact volume is defined as

$$\Delta V = V - V_0$$

If we let $f(r) = \frac{4\pi}{3} r^3$, note that this error can be written as

$$\Delta V = f(r) - f(r_0)$$
Using a linear approximation (i.e., tangent line approximation), this error is estimated as

$$\Delta V \approx f'(r_0) \Delta r$$

where we have defined $\Delta r = r - r_0$, which is the error in the radius. Calculating the derivative gives

$$\Delta V \approx 4\pi r_0^2 \Delta r$$

We are interested in the relative error in the volume, which is $\frac{\Delta V}{V_0}$. So dividing our approximation by $V$ gives the following.

$$\frac{\Delta V}{V_0} \approx \frac{4\pi r_0^2 \Delta r}{V_0} = \frac{4\pi r_0^2 \Delta r}{\frac{4\pi}{3} r_0^3} = 3 \frac{\Delta r}{r_0}$$

Observe that $\frac{\Delta r}{r_0}$ is the relative error in the radius (which is 1% for this problem). Hence we have found that the relative error in the volume is about 3 times as large as the relative error in the radius. So the relative error in the volume is about 3%.