1. A rock is dropped into a lake and an expanding circular ripple results. When the radius of the ripple is 8 inches, the radius is increasing at a rate of 3 inches per second. At what rate is the area enclosed by the ripple changing at this time?

**Solution**
Let $r$ be the radius of the ripple. At any time, the area $A$ enclosed by the ripple is given by

$$A = \pi r^2$$

Differentiating with respect to time gives us the following equation, which also holds for all time.

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

We now substitute the information relevant to the desired specific time. We substitute $r = 8$ and $\frac{dr}{dt} = 3$ into both equations.

$$A = 64\pi$$

$$\frac{dA}{dt} = 48\pi$$

Hence the area is increasing at a rate of $48\pi$ in$^2$/s.

2. An environmental study of a certain community indicates that there will be

$$Q(p) = 2p^2 + 6p + 1$$

units of a harmful pollutant in the air when the population is $p$ thousand. The population is currently 30,000 and is increasing at a rate of of 2,000 per year. At what rate is the level of the air pollution increasing currently?

**Solution**
Differentiating our equation relating $Q$ and $p$ with respect to time gives

$$\frac{dQ}{dt} = (4p + 6) \frac{dp}{dt}$$

We now substitute the information relevant to the desired specific time. We substitute $p = 30$ and $\frac{dp}{dt} = 2$ into both equations.

$$Q = 2(30)^2 + 6(30) + 1 = 1981$$

$$\frac{dQ}{dt} = (4 \cdot 30 + 6) \cdot 2 = 252$$

Hence the pollutant is increasing at a rate of 252 units per year.

3. Every day, a flight to Los Angeles flies directly over a man’s home at a constant altitude of 4 miles. If we assume that the plane is flying at a constant speed of 400 miles per hour, at what rate is the angle of elevation of the man’s line of sight changing with respect to time when the horizontal distance between the approaching plane and the man’s location is exactly 3 miles?
Solution
Let \( x \) be the horizontal distance from the man to the airplane. Let \( \theta \) be the angle of elevation. Since the height of the airplane is 4 miles, \( x \) and \( \theta \) satisfy the equation

\[
\tan(\theta) = \frac{4}{x}
\]

See the figure below. (Note that the diagram on the right shows a specific time. The diagram on the left shows a general time.)

Differentiating with respect to time gives

\[
\sec^2(\theta) \frac{d\theta}{dt} = -\frac{4}{x^2} \frac{dx}{dt}
\]

Now we substitute the information relevant to the desired specific time. We substitute \( \frac{dx}{dt} = -400 \) (negative because the distance \( x \) is decreasing since the plane is approaching the man) and \( x = 3 \).

\[
\tan(\theta) = \frac{4}{3}
\]

\[
\sec^2(\theta) \frac{d\theta}{dt} = \frac{1600}{9}
\]

We want an exact answer, so we use the identity \( \sec^2(\theta) = \tan^2(\theta) + 1 \). Given \( \tan(\theta) = \frac{4}{3} \), we obtain \( \sec^2(\theta) = \frac{25}{9} \). Hence our second equation above becomes

\[
\frac{25}{9} \frac{d\theta}{dt} = \frac{1600}{9}
\]

Solving for \( \frac{d\theta}{dt} \) gives

\[
\frac{d\theta}{dt} = \frac{1600}{25} = 64
\]

Hence the angle of elevation is increasing at a rate of 64 radians per hour.
4. A person 6 feet tall stands 10 feet from point $P$, which is directly beneath a lantern hanging 30 feet above the ground. The lantern starts to fall, thus causing the person’s shadow to lengthen. Given that the lantern falls $16t^2$ feet after $t$ seconds, how fast will the shadow be lengthening exactly 1 second after the lantern has started to fall?

Solution

Let $L$ be the length of the man’s shadow and let $s$ be the vertical distance from the lantern to the ground (point $P$). Using similar triangles, we see that $L$ and $s$ satisfy the equation

$\frac{s}{6} = \frac{L + 10}{L}$

(The distances 6 and 10 are constant since they represent the height of the man and the horizontal distance from the man to the lantern, respectively.) See the figure below. (Note that the diagram on the right shows a specific time. The diagram on the left shows a general time.)

If $h$ is the distance the lantern has already fallen then $s + h = 30$ and $h = 16t^2$. So $s = 30 - 16t^2$. Substituting $s = 30 - 16t^2$ into our previous equation and simplifying gives us the following equation that is true for all time.

$5 - \frac{8}{3}t^2 = 1 + \frac{10}{L}$

Differentiating with respect to time gives

$-\frac{16}{3}t = -\frac{10}{L^2} \frac{dL}{dt}$

Now we substitute the information relevant to the desired specific time. We substitute $t = 1$.

$5 - \frac{8}{3} = 1 + \frac{10}{L}$

$-\frac{16}{3} = -\frac{10}{L^2} \frac{dL}{dt}$

The first equation gives $L = \frac{30}{4} = 7.5$. Substituting $L = 7.5$ into the second equation gives

$-\frac{16}{3} = -\frac{10}{(15/2)^2} \frac{dL}{dt}$

Solving for $\frac{dL}{dt}$ gives $\frac{dL}{dt} = 30$. Hence the man’s shadow is increasing at a rate of 30 feet per second.
...Alternatively, we can solve for \( L \) directly from the equation

\[
5 - \frac{8}{3}t^2 = 1 + \frac{10}{L}
\]

to obtain

\[
L = \frac{15}{6 - 4t^2}
\]

Differentiating with respect to time then gives

\[
\frac{dL}{dt} = \frac{120t}{(6 - 4t^2)^2}
\]

Substituting \( t = 1 \) then gives

\[
\frac{dL}{dt} = \frac{120}{(6 - 4)^2} = \frac{120}{4} = 30
\]

We recover the same answer using the previous method.

5. The volume of a spherical balloon is increasing at constant rate of \( 3 \text{ in}^3/\text{s} \). At what rate is the radius of the balloon changing when the radius is 2 in.?

**Solution**

Let \( r \) be the radius of the balloon and let \( V \) be the volume of the balloon. Then \( r \) and \( V \) satisfy the equation

\[
V = \frac{4\pi}{3}r^3
\]

Differentiating with respect to time gives

\[
\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}
\]

Now we substitute the information relevant to the desired specific time. We substitute \( r = 2 \) and \( \frac{dV}{dt} = 3 \).

\[
V = \frac{32\pi}{3}
\]

\[
3 = 16\pi \frac{dr}{dt}
\]

Hence \( \frac{dr}{dt} = \frac{3}{16\pi} \). So the radius of the balloon is increasing at a rate of \( \frac{3}{16\pi} \) inches per second.

6. At noon, a ship sails due north from a point \( P \) at 8 knots (nautical miles per hour). Another ship, sailing at 12 knots, leaves the same point 1 hour later on a course due east. How fast is the distance between the ships increasing at 2:00 PM?

**Solution**

Let \( y \) be the distance from \( P \) to the ship sailing north and let \( x \) be the distance from \( P \) to the ship sailing east. If \( \ell \) is the direct distance between the two ships, then Pythagorean theorem

\[
\ell^2 = x^2 + y^2
\]

Differentiating with respect to time gives

\[
2\ell \frac{d\ell}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}
\]

Now we substitute the information relevant to the desired specific time. We substitute \( x = 8t \), \( y = 12(t-1) \), \( \frac{dx}{dt} = 8 \), and \( \frac{dy}{dt} = 12 \).

\[
\ell = \sqrt{64t^2 + 144(t-1)^2}
\]

Differentiating with respect to time gives

\[
\frac{d\ell}{dt} = \frac{80t}{\sqrt{64t^2 + 144(t-1)^2}} + \frac{144(t-1)}{\sqrt{64t^2 + 144(t-1)^2}}
\]

Substituting \( t = 2 \) then gives

\[
\frac{d\ell}{dt} = \frac{80(2)}{\sqrt{64(2)^2 + 144(2-1)^2}} + \frac{144(2-1)}{\sqrt{64(2)^2 + 144(2-1)^2}} = \frac{160}{16\sqrt{16 + 144}} + \frac{144}{16\sqrt{16 + 144}} = \frac{304}{32\sqrt{160}} = \frac{304}{640} = 0.48
\]

So the distance between the ships is increasing at a rate of approximately 0.48 knots per hour at 2:00 PM.
shows that $x$, $y$, and $\ell$ satisfy the equation

$$x^2 + y^2 = \ell^2$$

Differentiating with respect to time gives (after canceling a common factor of 2)

$$\frac{dx}{dt} + \frac{dy}{dt} \frac{d\ell}{dt} = \ell \frac{d\ell}{dt}$$

Now we substitute the information relevant to the desired specific time. At 2:00 PM, the northbound ship is at a distance of $y = 8 \cdot 2 = 16$ nautical miles. At the same time, the eastbound ship is at a distance of $x = 12 \cdot 1 = 12$ nautical miles. (Note that the northbound ship has been traveling for 2 hours, but the eastbound ship has been traveling only for 1 hour.) So into our equations we substitute $x = 12$, $y = 16$, $\frac{dx}{dt} = 12$, and $\frac{dy}{dt} = 8$.

$$12^2 + 16^2 = \ell^2$$
$$12 \cdot 12 + 16 \cdot 8 = \ell \frac{d\ell}{dt}$$

Solving for $\ell$ in the first equation gives $\ell = 20$. Substituting $\ell = 20$ into the second equation gives

$$144 + 128 = 20 \frac{d\ell}{dt}$$

Hence $\frac{d\ell}{dt} = \frac{68}{5} = 13.6$. The distance between the ships is increasing at a rate of 13.6 nautical miles per hour (or 13.6 knots).

7. Recall that a baseball diamond is a square of side length 90 ft. The corners of the diamond are labeled, in anti-clockwise order, home plate, first base, second base, and third base. Player A runs from home plate to first base at a speed of 20 ft/s. How fast is the player’s distance from second base changing when the player is halfway to first base?

**Solution**

![Diagram of baseball diamond]

The bases are labeled $H$, $B_1$, $B_2$, and $B_3$, in order. The player is at point $P$. The current distance from home plate to the player is $x$ and the current distance from the player to second base is $y$. Since each side of the square is 90 feet long, Pythagorean theorem gives us

$$y^2 = 90^2 + (90 - x)^2$$

(1)
Differentiating with respect to time gives

\[ 2y \frac{dy}{dt} = -2(90 - x) \frac{dx}{dt} \]  

(2)

We are interested in the time when the player is halfway to first base and we know the player runs at a constant speed of 20 ft/s. So we substitute \( \frac{dx}{dt} = 20 \) and \( x = 45 \) into equations (1) and (2).

\[ y^2 = 90^2 + 45^2 \]

\[ 2y \frac{dy}{dt} = -1800 \]

Solving these equations simultaneously for \( \frac{dy}{dt} \) gives

\[ \frac{dy}{dt} = -\frac{1800}{\sqrt{90^2 + 45^2}} = -4\sqrt{5} \text{ ft/s} \]

8. A particle moves along the elliptical path given by \( x^2 + 9y^2 = 13 \) in such a way that when it is at the point \((-2, 1)\), its \( x \)-coordinate is decreasing at the rate of 7 units per second. How fast is the \( y \)-coordinate changing at that instant?

**Solution**

Differentiating the equation \( x^2 + 9y^2 = 13 \) with respect to time gives

\[ 2x \frac{dx}{dt} + 18y \frac{dy}{dt} = 0 \]

Now we substitute \( x = -2, y = 1, \) and \( \frac{dx}{dt} = -7 \) to get the following.

\[ 28 + 18 \frac{dy}{dt} = 0 \]

Hence we find \( \frac{dy}{dt} = -14/9 \) units per second. That is, the \( y \)-coordinate is decreasing at a rate of 14/9 units per second.