1. For each of the following parts, calculate \( \frac{dy}{dx} \).

If \( y \) is given as an explicit function of \( x \), then the derivative must also be an explicit function of \( x \).

(a) \( x^2 + y^3 = 12 \)

(b) \( y + \frac{1}{xy} = x^2 \)

(c) \( y = \sqrt[3]{(x^{10} + 1)^3 (x^7 - 3)^8} \)

(d) \( y = \frac{e^{3x^2}}{(x^2 + 1)^2 (4x - 7)^{-2}} \)

(e) \( y = x^{\ln(\sqrt{3})} \)

(f) \( \sin(x + y) = x + \cos(y) \)

(g) \( \ln \left( \frac{x - y}{xy} \right) = \frac{1}{y} \)

(h) \( 6x^2 + 3xy + 2y^2 + 17y = 6 \)

Solution

Throughout the solution, \( y' \) will denote \( \frac{dy}{dx} \).

(a) Differentiating both sides with respect to \( x \) gives

\[ 2x + 3y^2 \cdot y' = 0 \]

Solving for \( y' \) gives

\[ y' = -\frac{2x}{3y^2} \]

(b) First write \( \frac{1}{xy} = x^{-1}y^{-1} \). Differentiating both sides with respect to \( x \) gives

\[ y' + x^{-1} \cdot (-1)y^{-2}y' + y^{-1} \cdot (-1)x^{-2} = 2x \]

Solving for \( y' \) gives

\[ y' = \frac{2x + x^{-2}y^{-1}}{1 - x^{-1}y^{-2}} \]

(c) We use logarithmic differentiation. Start by taking logarithms of both sides of the equation and simplifying.

\[
\ln(y) = \ln \left( \sqrt[3]{(x^{10} + 1)^3 (x^7 - 3)^8} \right) \\
= \ln \left( (x^{10} + 1)^{1/6} (x^7 - 3)^{4/9} \right) \\
= \frac{1}{6} \ln(x^{10} + 1) + \frac{4}{9} \ln(x^7 - 3)
\]

Differentiating both sides with respect to \( x \) gives

\[
\frac{1}{y} \cdot y' = \frac{1}{6} \cdot \frac{10x^9}{x^{10} + 1} + \frac{4}{9} \cdot \frac{7x^6}{x^7 - 3}
\]

Solving for \( y' \) gives

\[
y' = y \left( \frac{1}{6} \cdot \frac{10x^9}{x^{10} + 1} + \frac{4}{9} \cdot \frac{7x^6}{x^7 - 3} \right)
\]
Replacing \( y \) with its explicit definition in terms of \( x \) gives

\[
y' = \sqrt[3]{(x^{10} + 1)^3 (x^7 - 3)^2} \left( \frac{1}{6} \cdot \frac{10x^9}{x^{10} + 1} + \frac{4}{9} \cdot \frac{7x^6}{x^7 - 3} \right)
\]

(d) We use logarithmic differentiation. Start by taking logarithms of both sides of the equation and simplifying.

\[
\ln(y) = \ln \left( \frac{e^{3x^2}}{(x^3 + 1)^2 (4x - 7)^{-2}} \right)
= \ln(e^{3x^2}) - \ln((x^3 + 1)^2) - \ln((4x - 7)^{-2})
= 3x^2 - 2 \ln(x^3 + 1) + 2 \ln(4x - 7)
\]

Differentiating both sides with respect to \( x \) gives

\[
\frac{1}{y} \cdot y' = 6x - 2 \cdot \frac{3x^2}{x^3 + 1} + 2 \cdot \frac{4}{4x - 7}
\]

Solving for \( y' \) gives

\[
y' = y \left( 6x - 2 \cdot \frac{3x^2}{x^3 + 1} + 2 \cdot \frac{4}{4x - 7} \right)
\]

Replacing \( y \) with its explicit definition in terms of \( x \) gives

\[
y' = \frac{e^{3x^2}}{(x^3 + 1)^2 (4x - 7)^{-2}} \left( 6x - 2 \cdot \frac{3x^2}{x^3 + 1} + 2 \cdot \frac{4}{4x - 7} \right)
\]

(e) We use logarithmic differentiation. Start by taking logarithms of both sides of the equation and simplifying.

\[
\ln(y) = \ln \left( x^{\ln(\sqrt{x})} \right)
= \ln(\sqrt{x}) \ln(x)
= \frac{1}{2} \ln(x) \ln(x)
= \frac{1}{2} (\ln(x))^2
\]

Differentiating both sides with respect to \( x \) gives

\[
\frac{1}{y} \cdot y' = \frac{1}{2} \cdot 2 \ln(x) \cdot \frac{1}{x} = \frac{\ln(x)}{x}
\]

Solving for \( y' \) gives

\[
y' = y \left( \frac{\ln(x)}{x} \right)
\]

Replacing \( y \) with its explicit definition in terms of \( x \) gives

\[
y' = x^{\ln(\sqrt{x})} \left( \frac{\ln(x)}{x} \right)
\]
(f) Differentiating both sides with respect to $x$ gives

$$\cos(x + y) \cdot (1 + y') = 1 - \sin(y)y'$$

Solving for $y'$ gives

$$y' = \frac{1 - \cos(x + y)}{\cos(x + y) + \sin(y)}$$

(g) First simplify the left side of the equation to make the differentiation easier.

$$\ln(x - y) - \ln(x) - \ln(y) = y^{-1}$$

Differentiating both sides of this equation with respect to $x$ gives

$$\frac{1}{x - y} \cdot (1 - y') - \frac{1}{x} \cdot y' = -y^{-2}y'$$

Now rewrite the equation with negative exponents to make solving for $y'$ easier.

$$(x - y)^{-1}(1 - y') - x^{-1} - y^{-1}y' = -y^{-2}y'$$

Solving for $y'$ gives

$$y' = \frac{(x - y)^{-1} - x^{-1}}{-y^{-2} + (x - y)^{-1} + y^{-1}}$$

(h) Differentiating both sides with respect to $x$ gives

$$12x + 3xy' + 3y + 4yy' + 17y' = 0$$

Solving for $y'$ gives

$$y' = \frac{-12x - 3y}{3x + 4y + 17}$$

2. Suppose $x^2 + y^2 = R^2$, where $R$ is a constant. Find $y''$ and fully simplify your answer as much as possible.

**Solution**

Differentiating both sides with respect to $x$ gives

$$2x + 2y \cdot y' = 0$$

Solving for $y'$ gives

$$y' = -\frac{x}{y}$$

Differentiating with respect to $x$ once more gives

$$y'' = -\frac{y - xy'}{y^2}$$
Now substitute $y' = -\frac{x}{y}$ and simplify.

$$y'' = -\frac{y - x \left(-\frac{x}{y}\right)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{R^2}{y^3}$$

3. Find an equation of the line tangent to the graph of

$$xe^y = 2xy + y^3$$

at the point \(\left(\frac{1}{e - 2}, 1\right)\).

**Solution**

Differentiating both sides with respect to \(x\) gives

$$xe^y \cdot y' + e^y = 2xy' + 2y + 3y^2 \cdot y'$$

Now substitute \(x = \frac{1}{e - 2}\) and \(y = 1\).

$$\frac{e}{e - 2} \cdot y' + e = \frac{2}{e - 2} \cdot y' + 2 + 3y'$$

Solving for \(y'\) gives

$$y' = \frac{e - 2}{2}$$

The desired tangent line has slope \(\frac{e - 2}{2}\) and passes through the point \(\left(\frac{1}{e - 2}, 1\right)\). Hence the tangent line is given by

$$y - 1 = \frac{e - 2}{2} \left( x - \frac{1}{e - 2} \right)$$

4. Find an equation of the line tangent to the graph of

$$\sin(x - y) = xy$$

at the point \((0, \pi)\).

**Solution**

Differentiating both sides with respect to \(x\) gives

$$\cos(x - y) \cdot (1 - y') = xy' + y$$

Now substitute \(x = 0\) and \(y = \pi\).

$$(-1)(1 - y') = \pi$$

Solving for \(y'\) gives

$$y' = \pi + 1$$
The desired tangent line has slope \( \pi + 1 \) and passes through the point \((0, \pi)\). Hence the tangent line is given by

\[
y - \pi = (\pi + 1)(x - 0)
\]

5. Suppose \( x \) and \( y \) satisfy the following equation.

\[
x^2 + xy + 3y^2 = 99
\]

(a) Find all points on the graph where the tangent line is horizontal.

(b) Find all points on the graph where the tangent line is vertical.

Solution

For both parts of the question, we need \( y' \). So first differentiate both sides with respect to \( x \).

\[
2x + xy' + y + 6yy' = 0
\]

Solving for \( y' \) gives

\[
y' = \frac{2x + y}{x + 6y}
\]

(a) The tangent line is horizontal where \( y' = 0 \). This means the numerator of \( y' \) must be equal to 0 and the denominator must be not equal to 0. Setting the numerator of \( y' \) equal to 0 gives the equation \( 2x + y = 0 \), or \( y = -2x \). Hence any point on the graph where the tangent line is horizontal must satisfy both the equation \( x^2 + x y + 3y^2 = 99 \) and \( y = -2x \). Substitution of the latter into the former gives

\[
99 = x^2 + x(-2x) + 3(-2x)^2 = 11x^2
\]

Solving for \( x \) gives \( x = \pm 3 \). Hence there are two points on the graph where the tangent line is horizontal.

\[
P_1 = (-3, 6) \quad P_2 = (3, -6)
\]

We may then verify that neither of these points causes the denominator of \( y' \) to be equal to 0.

(b) The tangent line is vertical where \( y' \) is infinite. This means the denominator of \( y' \) must be equal to 0 and the numerator must be not equal to 0. Setting the denominator of \( y' \) equal to 0 gives the equation \( x + 6y = 0 \), or \( x = -6y \). Hence any point on the graph where the tangent line is vertical must satisfy both the equation \( x^2 + x y + 3y^2 = 99 \) and \( x = -6y \). Substitution of the latter into the former gives

\[
99 = (-6y)^2 + (-6y)y + 3y^2 = 33y^2
\]
Solving for $y$ gives $y = \pm \sqrt{3}$. Hence there are two points on the graph where the tangent line is horizontal.

$$P_1 = (6\sqrt{3}, -\sqrt{3})$$
$$P_2 = (-6\sqrt{3}, \sqrt{3})$$

We may then verify that neither of these points causes the numerator of $y'$ to be equal to 0.