1. It is estimated that $t$ years from now, the population (in thousands of people) of a certain suburban community is modeled by the formula

$$ p(t) = 20 - \frac{6}{t+1} $$

A separate environmental study indicates that the average daily level of carbon monoxide in the air (measured in ppm) will be

$$ L(p) = 0.5\sqrt{p^2 + p + 58} $$

when the population is $p$ thousand. Find the rate at which the level of carbon monoxide will be changing with respect to time two years from now. (Make sure to indicate units in your answer.)

**Solution**

Note that $L$ is really a composition of functions. That is, $L = L(p(t))$. Hence the chain rule gives us the following.

$$ \frac{dL}{dt} = L'(p(t)) \cdot p'(t) $$

We are interested in the value of $\frac{dL}{dt}$ when $t = 2$. Substituting $t = 2$ thus gives us

$$ \left. \frac{dL}{dt} \right|_{t=2} = L'(p(2)) \cdot p'(2) = L'(18) \cdot p'(2) $$

(We have used the fact that $p(2) = 18$.) Now we compute derivatives.

$$ L'(p) = 0.5 \cdot \frac{1}{2} (p^2 + p + 58)^{-1/2} \cdot (2p + 1) = \frac{2p + 1}{4\sqrt{p^2 + p + 58}} $$

$$ p'(t) = -6(-1)(t+1)^{-2} \cdot (1) = \frac{6}{(t+1)^2} $$

For both derivatives, we have used chain rule. Hence we have $L'(18) = \frac{37}{80}$ and $p'(2) = \frac{2}{3}$. The desired rate (rate at which $L$ is changing with respect to $t$) is then

$$ \left. \frac{dL}{dt} \right|_{t=2} = \frac{37}{80} \cdot \frac{2}{3} = \frac{37}{120} $$

(The units of this rate are ppm per thousand people.)

2. Suppose $g$ and $h$ are differentiable functions. Selected values of $g$, $h$, and their derivatives are given below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
<th>$g'(x)$</th>
<th>$h(x)$</th>
<th>$h'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>7</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>-3</td>
<td>-9</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>16</td>
<td>5</td>
<td>-1</td>
<td>1</td>
<td>-6</td>
</tr>
</tbody>
</table>

Define the function $f$ by the formula

$$ f(x) = g(\sqrt{x}) h(x^2) $$
(a) Calculate \( f(4) \) or explain why there is not enough information to do so.

(b) Calculate \( f'(4) \) or explain why there is not enough information to do so.

\[
\begin{align*}
\text{Solution} \\
(a) \ f(4) &= g(\sqrt{4})h(4^2) = g(2)h(16) = 1 \cdot 1 = 1 \\
(b) \ f'(x) &= g'(\sqrt{x}) h'(x^2) \cdot 2x + h'(x^2) g'(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2} \\
&= 2x g'(\sqrt{x}) h'(x^2) + \frac{g'(\sqrt{x}) h'(x^2)}{2\sqrt{x}} \\
\] Now we substitute \( x = 4 \) and use the table values.
\[
f'(4) = 8g(2)h'(16) + \frac{g'(2)h(16)}{4} = 8 \cdot 1 \cdot (-6) + \frac{7 \cdot 1}{4} = - \frac{185}{4}
\]

3. Find the \( x \)-coordinate of each point at which the graph of \( y = f(x) \) has a horizontal tangent line.

(a) \( f(x) = (2x^2 - 7)^3 \)  
(b) \( f(x) = x^2 e^{1-3x} \)  
(c) \( f(x) = \ln(3x^4 + 6x^2 - 4x^3 - 12x + 6) \)  
(d) \( f(x) = \frac{(e^{3x} + e^{-3x})^2}{e^{3x}} \)

\[
\text{Solution} \\
(a) \text{ Horizontal lines have slope 0 and the slope of the tangent line is given by the derivative.} \\
\text{Hence we must solve the equation } f'(x) = 0. \text{ Computing the derivative requires chain rule.} \\
\begin{align*}
f'(x) &= 3(2x^2 - 7)^2 \cdot (4x) \\
&= 12x(2x^2 - 7)^2 \\
\text{Hence either } 12x = 0 \text{ (whence } x = 0) \text{ or } 2x^2 - 7 = 0 \text{ (whence } x = -\sqrt{\frac{7}{2}} \text{ or } x = \sqrt{\frac{7}{2}}). \\
\end{align*}
(b) \text{ Horizontal lines have slope 0 and the slope of the tangent line is given by the derivative.} \\
\text{Hence we must solve the equation } f'(x) = 0. \text{ Computing the derivative requires chain rule and product rule.} \\
\begin{align*}
f'(x) &= x^2 e^{1-3x} \cdot (-3) + 2x \cdot e^{1-3x} \\
&= e^{1-3x}(-3x^2 + 2x) \\
&= -3x^2 + 2x = x(-3x + 2) \\
\text{Hence either } x = 0 \text{ or } -3x + 2 = 0 \text{ (whence } x = \frac{2}{3}). \\
\end{align*}
(c) \text{ Horizontal lines have slope 0 and the slope of the tangent line is given by the derivative.} \\
\text{Hence we must solve the equation } f'(x) = 0. \text{ Computing the derivative requires chain rule.}
f′(x) = \frac{1}{3x^4 + 6x^2 - 4x^3 - 12x + 6} \cdot (12x^3 + 12x - 12x^2 - 12)

0 = \frac{12x^3 + 12x - 12x^2 - 12}{3x^4 + 6x^2 - 4x^3 - 12x + 6}
0 = 12x^3 + 12x - 12x^2 - 12
0 = x^3 + x - x^2 - 1
0 = x(x^2 + 1) - (x^2 + 1) = (x - 1)(x^2 + 1)

Hence either x − 1 = 0 (whence x = 1). (The equation x^2 + 1 = 0 has no solutions.) However, we see that x = 1 is not in the domain of f, since the argument of a logarithm must be a strictly positive number. (Attempting to substitute x = 1 into f(x) gives the expression ln(−1).) So there are no points where the tangent line is horizontal.

(d) Horizontal lines have slope 0 and the slope of the tangent line is given by the derivative. Hence we must solve the equation f′(x) = 0. Before computing the derivative, we will simplify the function a bit. Combining all terms under one squaring operation gives the following.

\[ f(x) = \left( e^{3x} + e^{-3x} \right)^2 = \left( \frac{e^{3x} + e^{-3x}}{e^{3x/2}} \right)^2 = \left( \frac{e^{3x} + e^{-3x}}{e^{3x/2}} \right)^2 = \left( e^{3x/2} + e^{-9x/2} \right)^2 \]

Computing the derivative now requires just chain rule.

\[ f′(x) = 2 \left( e^{3x/2} + e^{-9x/2} \right) \cdot \left( \frac{3}{2} e^{3x/2} - \frac{9}{2} e^{-9x/2} \right) \]

Now we solve f′(x) = 0, which gives us two equations to solve. The first equation, e^{3x/2} + e^{-9x/2} = 0, has no solution since e^z is always positive for any z, and so the left-hand side of the equation is the sum of two positive terms (and hence can’t equal 0). The second equation we get from f′(x) = 0 is the following.

\[ 0 = \frac{3}{2} e^{3x/2} - \frac{9}{2} e^{-9x/2} \]
\[ 0 = e^{3x/2} - 3e^{-9x/2} \]
\[ 3e^{-9x/2} = e^{3x/2} \]
\[ 3 = e^{6x} \]
\[ \ln(3) = 6x \]
\[ \frac{1}{6} \ln(3) = x \]

Hence the only horizontal tangent line occurs at x = \( \frac{1}{6} \ln(3) \).

4. Calculate f′(x) for each function below. After computing the derivative, do not simplify your answer.

(a) f(x) = \sqrt{\sin(x)}
(b) f(x) = \sin(\sqrt{x})
(c) f(x) = \sqrt{\sin(\sqrt{x})}
(d) f(x) = (x^3 - 3x + 2)^2
(e) f(x) = \frac{1}{(3x + 1)^2}
(f) f(x) = (2x + \sec(x))^2
(g) f(x) = e^{-2x} \sin(x)
(h) \( f(x) = \frac{\ln(2x + 1)}{(2x + 1)^2} \)

(i) \( f(x) = (\tan(x) + 1)^4 \cos(2x) \)

(j) \( f(x) = \left( \frac{6}{9 - 2x} \right)^8 \)

(k) \( f(x) = (\sin ((4x - 5)^2))^4 \)

(Many authors will write this function as \( f(x) = \sin^4(4x - 5)^2 \), despite the ambiguous and inconsistent notation.)

(l) \( f(x) = \sqrt{\sin(x) \cos(x)} \)

(m) \( f(x) = \sqrt{\frac{x^2 - 1}{x^3 + x}} \)

(n) \( f(x) = \ln(\ln(x)) \)

(o) \( f(x) = \sin(\sin(\sin(x))) \)

(p) \( f(x) = \left( x + (x + \sin(x)^2)^3 \right)^4 \)

(q) \( f(x) = |x| \)

(Hint: use the identity \(|x| = \sqrt{x^2}\).)

**Solution**

(a) \( f'(x) = \frac{1}{2} (\sin(x))^{-1/2} \cos(x) \)

(b) \( f'(x) = \cos(\sqrt{x}) \cdot \frac{1}{2} x^{-1/2} \)

(c) \( f'(x) = \frac{1}{2} (\sin(x))^{-1/2} \cos(\sqrt{x}) \cdot \frac{1}{2} x^{-1/2} \)

(d) \( f'(x) = 2(x^3 - 3x + 2)(3x^2 - 3) \)

(e) \( f'(x) = -2(3x + 1)^{-3} \cdot 3 \)

(f) \( f'(x) = 2(2x + \sec(x))(2 + \sec(x) \tan(x)) \)

(g) \( f'(x) = e^{-2x} \cos(x) - 2e^{-2x} \sin(x) \)

(h) \( f'(x) = \frac{(2x + 1)^2 \cdot \frac{1}{x+1} \cdot 2 - \ln(2x + 1) \cdot 2(2x + 1) \cdot 2}{(2x + 1)^4} \)

(i) \( f'(x) = (\tan(x) + 1)^4 (-\sin(2x)) \cdot 2 + \cos(2x) \cdot 4 (\tan(x) + 1)^3 \cdot \sec(x)^2 \)

(j) \( f'(x) = 6^8 \cdot (-8) \cdot (9 - 2x)^{-9} \cdot (-2) \)

(k) \( f'(x) = 4 \left( \sin ((4x - 5)^2) \right)^3 \cdot \cos ((4x - 5)^2) \cdot 2(4x - 5) \cdot 4 \)

(l) \( f'(x) = \frac{1}{3} (\sin(x) \cos(x))^{-2/3} \cdot (\sin(x)(-\sin(x)) + \cos(x) \cos(x)) \)

(m) \( f'(x) = \frac{1}{2} \left( \frac{x^2 - 1}{x^3 + x} \right)^{-1/2} \cdot \frac{(x^3 + x)(2x) - (x^2 - 1)(3x^2 + 1)}{(x^3 + x)^2} \)

(n) \( f'(x) = \frac{1}{\ln(x)} \cdot \frac{1}{x} \)

(o) \( f'(x) = \cos(\sin(\sin(x))) \cos(\sin(x)) \cos(x) \)

(p) \( f'(x) = 4 \left( x + (x + \sin(x)^2)^3 \right)^3 \cdot \left( 1 + 3(x + \sin(x)^2)^2 \cdot (1 + 2 \sin(x) \cos(x)) \right) \)

(q) \( f'(x) = \frac{1}{2} (x^2)^{-1/2} \cdot (2x) = \frac{x}{|x|} \)