1. Suppose the line described by \( y = 5x - 9 \) is tangent to the graph of \( y = f(x) \) at \( x = 4 \).

(a) Calculate \( f(4) \). If there is not enough information to do so, explain why.
(b) Calculate \( f(3) \). If there is not enough information to do so, explain why.
(c) Calculate \( f'(4) \). If there is not enough information to do so, explain why.
(d) Calculate \( f'(3) \). If there is not enough information to do so, explain why.

Solution

(a) The tangent line at \( x = a \) is defined to be the line to pass through the point \((a, f(a))\) with slope \( f'(a) \). The line \( y5x - 9 \) passes through \((4, 11)\) and is tangent to the graph of \( y = f(x) \) at \( x = 4 \). Hence \( f(4) = 11 \).

(b) The tangent line at \( x = 4 \) has no relation to the function \( f(x) \) at any other value of \( x \). So there is not enough information to tell the value of \( f(3) \).

(c) See solution for part (a). The slope of the line \( y = 5x - 9 \) is 5, whence \( f'(4) = 5 \).

(d) See solution for part (b). There is not enough information to tell the value of \( f'(3) \).

2. Use the limit definition of the derivative to calculate the derivative of \( f \) at \( x = 5 \). Then find an equation for the line tangent to the graph of \( y = f(x) \) at \( x = 5 \).

(a) \( f(x) = 2x - 1 \)
(b) \( f(x) = (2x - 1)^2 \)
(c) \( f(x) = \sqrt{2x - 1} \)
(d) \( f(x) = \frac{1}{2x - 1} \)
(e) \( f(x) = \frac{1}{\sqrt{2x - 1}} \)
(f) \( f(x) = \frac{1}{\sqrt{2x - 1}} \)

Solution

(a) Observe that \( f(5) = 9 \). Then, by definition, we have the following.

\[
f'(5) = \lim_{h \to 0} \left( \frac{f(5 + h) - f(5)}{h} \right) = \lim_{h \to 0} \left( \frac{2(5 + h) - 1 - 9}{h} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{2h}{h} \right) = \lim_{h \to 0} (2) = 2
\]

Hence the tangent line has equation \( y - 9 = 2(x - 5) \).

(b) Observe that \( f(5) = 81 \). Then, by definition, we have the following.

\[
f'(5) = \lim_{h \to 0} \left( \frac{f(5 + h) - f(5)}{h} \right) = \lim_{h \to 0} \left( \frac{(2(5 + h) - 1)^2 - 81}{h} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{(2h + 9)^2 - 81}{h} \right) = \lim_{h \to 0} \left( \frac{4h^2 + 36h + 81 - 81}{h} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{4h^2 + 36h}{h} \right) = \lim_{h \to 0} (4h + 36) = 36
\]

Hence the tangent line has equation \( y - 81 = 36(x - 5) \).
(c) Observe that \( f(5) = 3 \). Then, by definition, we have the following.

\[
f'(5) = \lim_{h \to 0} \left( \frac{f(5 + h) - f(5)}{h} \right) = \lim_{h \to 0} \left( \frac{\sqrt{2(5 + h)} - 1 - 3}{h} \right)
\]
\[
= \lim_{h \to 0} \left( \frac{\sqrt{2h + 9} - 3}{h} \right) = \lim_{h \to 0} \left( \frac{2h + 9 - 9}{h(\sqrt{2h + 9} + 3)} \right)
\]
\[
= \lim_{h \to 0} \left( \frac{2}{\sqrt{2h + 9} + 3} \right) = \frac{2}{\sqrt{9} + 3} = \frac{1}{3}
\]

Hence the tangent line has equation \( y - 3 = \frac{1}{3}(x - 5) \).

(d) Observe that \( f(5) = \frac{1}{9} \). Then, by definition, we have the following.

\[
f'(5) = \lim_{h \to 0} \left( \frac{f(5 + h) - f(5)}{h} \right) = \lim_{h \to 0} \left( \frac{\frac{1}{2(5 + h) - 1} - \frac{1}{9}}{h} \right)
\]
\[
= \lim_{h \to 0} \left( \frac{\frac{1}{2h + 9} - \frac{1}{9}}{h} \right) = \lim_{h \to 0} \left( \frac{9 - (2h + 9)}{9h(2h + 9)} \right)
\]
\[
= \lim_{h \to 0} \left( \frac{-2}{9(2h + 9)} \right) = \frac{-2}{9(0 + 9)} = -\frac{2}{81}
\]

Hence the tangent line has equation \( y - \frac{1}{9} = -\frac{2}{81}(x - 5) \).

(e) Observe that \( f(5) = \frac{1}{3} \). Then, by definition, we have the following.

\[
f'(5) = \lim_{h \to 0} \left( \frac{f(5 + h) - f(5)}{h} \right) = \lim_{h \to 0} \left( \frac{\frac{1}{\sqrt{2(5 + h)} - 1} - \frac{1}{3}}{h} \right)
\]
\[
= \lim_{h \to 0} \left( \frac{\sqrt{2h + 9} - \frac{1}{3}}{h} \right) = \lim_{h \to 0} \left( \frac{3 - \sqrt{2h + 9}}{3h\sqrt{2h + 9}} \right)
\]
\[
= \lim_{h \to 0} \left( \frac{9 - (2h + 9)}{3h\sqrt{2h + 9} + (3 + \sqrt{2h + 9})} \right)
\]
\[
= \lim_{h \to 0} \left( \frac{-2}{3\sqrt{2h + 9} + (3 + \sqrt{2h + 9})} \right) = \frac{-2}{3\sqrt{9}(3 + \sqrt{9})} = -\frac{1}{27}
\]

Hence the tangent line has equation \( y - \frac{1}{3} = -\frac{1}{27}(x - 5) \).
(f) Observe that \( f(5) = \frac{1}{\sqrt{10} - 1} \). Then, by definition, we have the following.

\[
f'(5) = \lim_{h \to 0} \left( \frac{f(5 + h) - f(5)}{h} \right) = \lim_{h \to 0} \left( \frac{\frac{1}{\sqrt{2(5+h)+10} - 1} - \frac{1}{\sqrt{10}-1}}{h} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{\sqrt{10} - \sqrt{2h+10}}{h(\sqrt{10}-1)(\sqrt{2h+10} - 1)} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{10 - (2h+10)}{h(\sqrt{10}-1)(\sqrt{2h+10} + 10)} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{-2}{(\sqrt{10}-1)(\sqrt{2h+10} + 10)} \right)
\]

\[
= -\frac{2}{(\sqrt{10}-1)(\sqrt{10} + \sqrt{0+10})} = -\frac{1}{\sqrt{10}(\sqrt{10}-1)^2}
\]

Hence the tangent line has equation \( y - \frac{1}{\sqrt{10} - 1} = -\frac{1}{\sqrt{10}(\sqrt{10}-1)^2}(x - 5) \).

3. The graph of \( y = f(x) \) is given below. Sketch a graph of \( y = f'(x) \). Only the general shape is important. Do not worry about scales.

![Graph of f(x) and f'(x) combined](image)

**Solution**

The graph of \( y = f(x) \) is shown below in black. The graph of \( y = f'(x) \) is shown below in blue.
4. Consider the following function.

\[ f(x) = \begin{cases} 
-x^2, & x < 0 \\
-x^2 + 2x, & 0 \leq x < 1 \\
6x - x^2 + c, & x \geq 1 
\end{cases} \]

(a) Is \( f \) differentiable at \( x = 0 \)?

(b) Is there a value of \( c \) that makes \( f \) differentiable at \( x = 1 \)? If so, calculate it. If not, explain why.

Solution

(a) Observe that \( f(0) = 0 \). Then, by definition, we have the following.

\[
 f'(0) = \lim_{h \to 0} \left( \frac{f(0 + h) - f(0)}{h} \right) = \lim_{h \to 0} \left( \frac{f(h)}{h} \right)
\]

Since \( f(h) \) is piecewise defined and changes definition at \( h = 0 \), we must compute the left- and right-limits.

\[
\lim_{h \to 0^+} \left( \frac{f(h)}{h} \right) = \lim_{h \to 0^+} \left( \frac{-h^2}{h} \right) = \lim_{h \to 0^+} (-h) = 0
\]

\[
\lim_{h \to 0^-} \left( \frac{f(h)}{h} \right) = \lim_{h \to 0^-} \left( \frac{h^2 + 2h}{h} \right) = \lim_{h \to 0^-} (h + 2) = 2
\]

The one-sided limits are not equal, whence \( f'(0) \) does not exist. That is, \( f \) is not differentiable at \( x = 0 \).

(b) Recall that continuity is a necessary (but not sufficient) condition for differentiability. That is, if \( f \) is to be differentiable at \( x = 1 \), then \( f \) must also be continuous at \( x = 1 \). So first we determine the value of \( c \) that makes \( f \) continuous at \( x = 1 \).

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 2x) = 1 + 2 = 3
\]

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (6x - x^2 + c) = 6 - 1 + c = 5 + c
\]

\[
f(1) = (6x - x^2 + c) \bigg|_{x=1} = 6 - 1 + c = 5 + c
\]
So we must have that $3 = 5 + c$, or $c = -2$.

Now we must check whether this value $c$ makes $f$ differentiable at $x = 1$. Observe that with $c = -2$, we have $f(1) = 3$. So, by definition, we have the following.

$$f'(1) = \lim_{h \to 0} \left( \frac{f(1 + h) - f(1)}{h} \right) = \lim_{h \to 0} \left( \frac{f(1 + h) - 3}{h} \right)$$

Since $f(1+h)$ is piecewise defined and changes definition at $h = 0$ (equivalently, at $x = 1$), we must compute the left- and right-limits.

$$\lim_{h \to 0^-} \left( \frac{f(1 + h) - 3}{h} \right) = \lim_{h \to 0^-} \left( \frac{(1 + h)^2 + 2(1 + h) - 3}{h} \right) = \lim_{h \to 0^-} \left( \frac{h^2 + 2h + 1 + 2 + 2h - 3}{h} \right) = \lim_{h \to 0^-} \left( \frac{h^2 + 4h}{h} \right) = \lim_{h \to 0^-} (h + 4) = 4$$

$$\lim_{h \to 0^+} \left( \frac{f(1 + h) - 3}{h} \right) = \lim_{h \to 0^+} \left( \frac{6(1 + h) - (1 + h)^2 - 2 - 3}{h} \right) = \lim_{h \to 0^+} \left( \frac{6 + 6h - (1 + 2h + h^2) - 2 - 3}{h} \right) = \lim_{h \to 0^+} \left( \frac{4h - h^2}{h} \right) = \lim_{h \to 0^+} (4 - h) = 4$$

The one-sided limits are equal, whence $f'(1) = 4$. That is, the choice of $c = -2$ makes $f$ differentiable at $x = 1$. 