1. Determine all points where the following function is continuous. 

Make sure you give a justification for any x-value at which you claim f is continuous.

\[ f(x) = \begin{cases} 
3x^2 - x + 1, & x < -2 \\
15 + \sin(2\pi x), & -2 \leq x < 3 \\
2x - 4, & 3 \leq x 
\end{cases} \]

**Solution**

Each individual piece is continuous for all real numbers, so we only have to check continuous at the transition points \( x = -2 \) and \( x = 3 \). To guarantee continuity at a point, the left-limit, right-limit, and function value must all be equal at that point.

- \( (x = -2) \):

  \[
  \lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (3x^2 - x + 1) = 3(-2)^2 - (-2) = 15 \\
  \lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (15 + \sin(2\pi x)) = 15 + \sin(-4\pi) = 15 \\
  f(-2) = (15 + \sin(2\pi x))|_{x=-2} = 15 + \sin(-4\pi) = 15
  \]

  Hence \( f \) is continuous at \( x = -2 \).

- \( (x = 3) \):

  \[
  \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (15 + \sin(2\pi x)) = 15 + \sin(-6\pi) = 15 \\
  \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2x - 4) = 2(3) - 4 = 2 \\
  f(3) = (2x - 4)|_{x=3} = 2(3) - 4 = 2
  \]

  Hence \( f \) is not continuous at \( x = 3 \).

The function \( f \) is continuous on \(( -\infty, 3 ) \cup (3, \infty) \).

2. Let \( f(x) = \frac{x^3 - 9x}{x + 3} \).

(a) What is the domain of \( f \)?

(b) Find all points where \( f \) is discontinuous.

(c) For each x-value you found in part (b), determine what value should be assigned to \( f \), if any, to guarantee that \( f \) will be continuous there.

  *(For example, if you claim \( f \) is discontinuous at \( x = a \), then you should determine the value that should be assigned to \( f(a) \), if any, to guarantee that \( f \) will be continuous at \( x = a \).)*

**Solution**

- (a) \(( -\infty, -3 ) \cup (-3, \infty) \).

- (b) Since \( f \) is a rational function, \( f \) is discontinuous only at points not in its domain. Hence \( f \) is discontinuous only at \( x = -3 \).

- (c) A function is continuous at a point if and only if its function value is equal to the limit
value there. Hence the only possible choice for \( f(-3) \) to make \( f \) continuous is

\[
f(-3) = \lim_{x \to -3} f(x) = \lim_{x \to -3} \left( \frac{x^3 - 9x}{x + 3} \right) = \lim_{x \to -3} \left( \frac{x(x - 3)(x + 3)}{x + 3} \right)
\]

\[
= \lim_{x \to -3} (x(x - 3)) = (-3)(-3 - 3) = 18
\]

Hence if \( f \) is to be continuous at \( x = -3 \), we must choose \( f(-3) = 18 \).

3. Let \( f(x) = \frac{\sqrt{2x^2 + 1} - 1}{x^2(x - 3)} \).

(a) What is the domain of \( f \)?

(b) Find all points where \( f \) is discontinuous.

(c) For each \( x \)-value you found in part (b), determine what value should be assigned to \( f \), if any, to guarantee that \( f \) will be continuous there.

(For example, if you claim \( f \) is discontinuous at \( x = a \), then you should determine the value that should be assigned to \( f(a) \), if any, to guarantee that \( f \) will be continuous at \( x = a \).)

Solution

(a) Note that \( 2x^2 + 1 \geq 0 \) always, so the only points not in the domain of \( f \) are those for which \( x^2(x - 3) = 0 \). Hence the domain is \((-\infty, 0) \cup (0, 3) \cup (3, \infty)\).

(b) Since \( f \) is an algebraic function, \( f \) is discontinuous only at points not in its domain. Hence \( f \) is discontinuous only at \( x = 0 \) and \( x = 3 \).

(c) A function is continuous at a point if and only if its function value is equal to the limit value there. Hence the only possible choice for \( f(0) \) to make \( f \) continuous there is

\[
f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \left( \frac{\sqrt{2x^2 + 1} - 1}{x^2(x - 3)} \right)
\]

\[
= \lim_{x \to 0} \left( \frac{\sqrt{2x^2 + 1} - 1}{x^2(x - 3)} \cdot \frac{\sqrt{2x^2 + 1} + 1}{\sqrt{2x^2 + 1} + 1} \right) = \lim_{x \to 0} \left( \frac{2x^2}{x^2(x - 3)(\sqrt{2x^2 + 1} + 1)} \right)
\]

\[
= \lim_{x \to 0} \left( \frac{2}{(x - 3)(\sqrt{2x^2 + 1} + 1)} \right) = \frac{2}{(-3)(1 + 1)} = -\frac{1}{3}
\]

Hence if \( f \) is to be continuous at \( x = 0 \), we must choose \( f(0) = -\frac{1}{3} \).

The only possible choice for \( f(3) \) to make \( f \) continuous there is

\[
f(3) = \lim_{x \to 3} f(x) = \lim_{x \to 3} \left( \frac{\sqrt{2x^2 + 1} - 1}{x^2(x - 3)} \right)
\]

Observe that substitution of \( x = 3 \) gives the undefined form \( \frac{\sqrt{19} - 1}{0} \), or a non-zero number divided by zero. This indicates that the left- and right-limits are both infinite. Hence the
overall limit is either infinite or does not exist. In any event, there is no value we may assign to \( f(3) \) to make \( f \) continuous at \( x = 3 \).

4. Find the values of the constants \( a \) and \( b \) that make \( f \) continuous for all real numbers.

\[
f(x) = \begin{cases} 
  ax^2 - x, & x < 4 \\
  6, & x = 4 \\
  x^3 + bx, & x > 4 
\end{cases}
\]

**Solution**

Any values of \( a \) and \( b \) make each individual piece continuous for all real numbers. Hence we need only force continuity at \( x = 4 \).

\[
\begin{align*}
\lim_{x \to 4^-} f(x) &= \lim_{x \to 4^-} (ax^2 - x) = 16a - 4 \\
\lim_{x \to 4^+} f(x) &= \lim_{x \to 4^+} (x^3 + bx) = 64 + 4b \\
f(4) &= 6
\end{align*}
\]

If \( f \) is to be continuous at \( x = 4 \), these three values must be equal. Hence we obtain the two equations \( 16a - 4 = 6 \) (whence \( a = \frac{10}{16} \)) and \( 64 + 4b = 6 \) (whence \( b = -\frac{29}{2} \)).

5. Find the values of the constants \( a \) and \( b \) that make \( f \) continuous at \( x = 0 \). You may assume \( a > 0 \).

\[
f(x) = \begin{cases} 
  \frac{1 - \cos(ax)}{x^2}, & x < 0 \\
  2a + b, & x = 0 \\
  \frac{x^2 - bx}{\sin(x)}, & x > 0 
\end{cases}
\]

**Solution**

We need only force continuity at \( x = 0 \).

\[
\begin{align*}
\lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} \left( \frac{1 - \cos(ax)}{x^2} \right) = \lim_{x \to 0^-} \left( \frac{1 - \cos(ax)}{x^2} \cdot \frac{1 + \cos(ax)}{1 + \cos(ax)} \right) \\
&= \lim_{x \to 0^-} \left( \frac{1 - \cos(ax)^2}{x^2(1 + \cos(ax))} \right) = \lim_{x \to 0^-} \left( \frac{\sin(ax)^2}{x^2(1 + \cos(ax))} \right) \\
&= \lim_{x \to 0^-} \left( \left( a \cdot \frac{\sin(ax)}{ax} \right)^2 \cdot \frac{1}{1 + \cos(ax)} \right) = (a \cdot 1)^2 \cdot \frac{1}{1 + 1} = \frac{a^2}{2}
\end{align*}
\]

\[
\begin{align*}
\lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \left( \frac{x^2 - bx}{\sin(x)} \right) = \lim_{x \to 0^+} \left( \frac{x}{\sin(x)} \cdot (x - b) \right) = 1 \cdot (0 - b) = -b \\
f(0) &= 2a + b
\end{align*}
\]
If \( f \) is to be continuous at \( x = 0 \), these three values must be equal. Hence we obtain the two equations \( \frac{a^2}{2} = 2a + b \) and \( -b = 2a + b \). Solving the second equation for \( b \) gives \( b = -a \). Substituting \( b = -a \) into the first equation gives \( \frac{a^2}{2} = 2a - a = a \). Dividing by \( a \) (which we are told is positive!) gives \( \frac{a}{2} = 1 \), or \( a = 2 \). Hence we must choose \( a = 2 \) and \( b = -2 \).

6. Prove that the equation \( \sqrt{x} + x^3 = 1 \) has a solution in the interval \([0, 1]\).

**Solution**

Let \( f(x) = \sqrt{x} + x^3 - 1 \). We need to show that there exists \( c \) in the interval \([0, 1]\) such that \( f(c) = 0 \). Observe that \( f(0) = -1 < 0 \) and \( f(1) = 1 > 0 \). Since \( f \) is continuous on \([0, 1]\) and 0 is between \(-1\) and \(1\), it follows by the intermediate value theorem, that such a value of \( c \) exists.

7. Prove that the equation \( x^4 + 3x^2 + 2 = 4x^3 + 8x \) has a solution.

**Solution**

Let \( f(x) = x^4 + 3x^2 + 2 - 4x^3 - 8x \). We need to show that there exists \( c \) such that \( f(c) = 0 \). Observe that \( f(0) = 2 > 0 \) and \( f(1) = -6 < 0 \). Since \( f \) is continuous on \([0, 1]\) and 0 is between \(-6\) and \(2\), it follows by the intermediate value theorem, that there exists \( c \) in the interval \([0, 1]\) such that \( f(c) = 0 \).