1. Consider the following function.

\[
f(x) = \begin{cases} 
  x^3 + 27, & x \leq -3 \\
  \frac{x + 3}{2 - \sqrt{1-x}}, & -3 < x < 1 \\
  4, & x = 1 \\
  x^2 + 2x - 1, & 1 < x 
\end{cases}
\]

(a) Find all points where \( f \) is discontinuous. Be sure to give a full justification here.

(b) For each \( x \)-value you found in part (a), determine what value should be assigned to \( f \), if any, to guarantee that \( f \) will be continuous there. Justify your answer.

(For example, if you claim \( f \) is discontinuous at \( x = a \), then you should determine the value that should be assigned to \( f(a) \), if any, to guarantee that \( f \) will be continuous at \( x = a \)).

Solution

(a) First note that \( x = -3 \) and \( x = 1 \) are suspicious points, and so we must check continuity there. For all other points, note that each piece individually is continuous on the given intervals. The first piece \( (x^3 + 27) \) and third piece \( (x^2 + 2x - 1) \) are continuous on all intervals because they are polynomials. The second piece \( \left( \frac{x + 3}{2 - \sqrt{1-x}} \right) \), however, is not continuous on all intervals. Instead, we must require that \( 0 \leq 1 - x \) (or \( x \leq 1 \)) and \( 2 - \sqrt{1-x} \neq 0 \) (or \( x \neq -3 \)). But both of these conditions are satisfied on the indicated interval \( (-3 < x \leq 1) \).

Now we check each suspicious point. To guarantee continuity at \( x = a \), the left-limit, right-limit, and function value must all be equal at \( x = a \).

- \( (x = -3) \):

\[
\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (x^3 + 27) = (-3)^3 + 27 = 0 \\
\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \left( \frac{x + 3}{2 - \sqrt{1-x}} \right) \\
= \lim_{x \to -3^+} \left( \frac{x + 3}{2 - \sqrt{1-x}} \cdot \frac{2 + \sqrt{1-x}}{2 + \sqrt{1-x}} \right) \\
= \lim_{x \to -3^+} \frac{(x + 3)(2 + \sqrt{1-x})}{4 - (1-x)} \\
= \lim_{x \to -3^+} \frac{(x + 3)(2 + \sqrt{1-x})}{x + 3} \\
= \lim_{x \to -3^+} (2 + \sqrt{1-x}) = 2 + \sqrt{1 - (-3)} = 4 \\
f(-3) = (x^3 + 27) \big|_{x=-3} = (-3)^3 + 27 = 0
\]

(Note that when calculating the right limit, we first rationalized the denom-
inator, then canceled common factors, and finally substituted $x = -3$.) Since these three numbers are not all equal, $f$ is discontinuous at $x = -3$.

• ($x = 1$):

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left( \frac{x + 3}{2 - \sqrt{1 - x}} \right) = \frac{1 + 3}{2 - \sqrt{1 - 1}} = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 2x - 1) = 1^2 + 2(1) - 1 = 2$$

$f(1) = 4$

Since these three numbers are not all equal, $f$ is discontinuous at $x = 1$.

In summary, we have found that $f$ is continuous for all real numbers except $x = -3$ and $x = 1$.

(b) Since the one-sided limits at $x = -3$ are not equal, the two-sided limit $\lim_{x \to -3} f(x)$ does not exist. Hence it is not possible to assign a value to $f(-3)$ to make $f$ continuous at $x = -3$.

The one-sided limits at $x = 1$ are equal, and so $\lim_{x \to 1^-} f(x) = 2$. Hence if we assign $f(1)$ the value of 2, then we would have $\lim_{x \to 1^+} f(x) = f(1)$, which means $f$ would be continuous at $x = 1$.

2. Find all real solutions to the following equation.

$$\log_2(x) + \log_2(x - 3) = 2$$

Solution

Combine the logarithms using the identity $\log_a(x) + \log_a(y) = \log_a(xy)$. Then undo the logarithms by exponentiation, and solve the resulting equation.

$$\log_2(x) + \log_2(x - 3) = 2$$

$$\log_2(x(x - 3)) = 2$$

$$x(x - 3) = 2^2$$

$$x^2 - 3x - 4 = 0$$

$$(x - 4)(x + 1) = 0$$

Hence the two candidate solutions are $x = 4$ and $x = -1$. Now check these candidates in the original equation.

The candidate $x = 4$ gives the purported equation

$$\log_2(4) + \log_2(1) = 2$$

Since $\log_2(4) = 2$ and $\log_2(1) = 0$, this is a true equation. Hence $x = 4$ is a solution.
The candidate $x = -1$ gives the purported equation

$$\log_2(-1) + \log_2(-4) = 2$$

This is nonsense since the domain of $\log_a(x)$ is strictly positive $x$. That is, we cannot compute the logarithm of a negative number. So $x = -1$ is not a solution.

In summary, the only solution to the given equation is $x = 4$. 