Review for Exam #2:
(Covers 3.6–3.8 and 4.3–4.7.)

Exam Format:
• 7 fill-in: no work, no partial credit, 5pts each
• 4 long response: show work, partial credit, 10pts each
• 1 graph analysis: no work, 10 parts, 2.5pts each

Ex. 1 (Fall 2017)

Calculate the limit or show it does not exist.

$$\lim_{x \to 0} \left( 1 - \sin(4x) \right)^{6/x}$$

Solution:
D.S. of $x = 0$ gives the expression "1^\infty", which is indeterminate.

$$L = \lim_{x \to 0} \left( 1 - \sin(4x) \right)^{6/x}$$

$$\ln(L) = \lim_{x \to 0} \ln \left[ \left( 1 - \sin(4x) \right)^{6/x} \right]$$
\[ \ln(L) = \lim_{x \to 0} \left( \frac{6 \ln (1 - \sin(4x))}{x} \right) \]

\[ \ln(L) = \lim_{x \to 0} \left( \frac{6 \cdot \frac{1}{1 - \sin(4x)} (-\cos(4x))}{4} \right) \]

\[ \ln(L) = 6 \cdot \frac{1}{1 - 0} (-1) \cdot 4 = -24 \]

So \( \ln(L) = -24 \). So \( L = e^{-24} \).

**Ex. 2** (Fall 2018)

Find the largest area of a rectangle whose base is on \( x \)-axis and upper vertices are on graph of \( y = e^{-x^2/12} \).
Solution:

Objective: \( A(x, y) = 2xy \)

Constraint: \( y = e^{-x^2/12} \)

Substituting constraint into objective gives

\[
A(x, e^{-x^2/12}) = 2x e^{-x^2/12}
\]

Goal: Find the absolute maximum value of the function

\[ f(x) = 2x e^{-x^2/12} \]

on the interval \([0, \infty)\).
Find critical #'s:

• $f'(x)$ due: none

• $f'(x) = 0$:

$$f'(x) = 2x \cdot e^{-x^2/12} \cdot \left( -\frac{x}{6} \right) + e^{-x^2/12} \cdot 2$$

$$= e^{-x^2/12} \left( -\frac{x^2}{3} + 2 \right)$$

$f'(x) = 0 \implies$

$$e^{-x^2/12} = 0 \text{ or } -\frac{x^2}{3} + 2 = 0$$

no solutions

$$x = -\sqrt{6} \text{ or } x = \sqrt{6}$$

not in $[0, \infty)$

Now use first derivative test to determine maximum.

$$\begin{array}{c}
0 \quad + \quad \sqrt{6} \quad - \\
\hline
\end{array}$$

shape off

1 = $\sqrt{1}$

$\sqrt{9} = 3$ test point
\[ f'(x) = e^{-x^2/12} \left( -\frac{x^2}{3} + 2 \right) \]

\[ f'(1) = e^{-1/12} \left( -\frac{1}{3} + 2 \right) = e^{-1/12} \frac{5}{3} \approx 1.477 \]

\[ f'(3) = e^{-3^2/12} \left( -\frac{9}{3} + 2 \right) = e^{-1/4} \frac{-7}{3} \approx -0.815 \]

So the max of \( f(x) \) on \((0, \infty)\) occurs at \( x = \sqrt{6} \). That max area is \( f(\sqrt{6}) = 2\sqrt{6} e^{-1/2} = 2\sqrt{6} \frac{1}{e} \).

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**Ex. 3** (Spring 2018)

The daily output is

\[ Q(L) = 1500 L^{2/3} \]

where \( L \) is the size of the labor force, measured in worker-hours. Currently 1000 worker-hours of labor are used each day. Use a linear approximation to estimate the effect on daily output if the labor force is cut to 975 worker-hours.
Solution:
We first find the tangent line to \( y = Q(L) \) at \( L = 1000 \).

Point: \( Q(1000) = 1500 \cdot 1000^{2/3} \)
= \( 1500 \cdot 100 \)
= \( 150,000 \)

Slope: \( Q'(L) = 1500 \cdot \frac{2}{3} \cdot L^{-1/3} \)
\( Q'(1000) = 1000 \cdot 1000^{-1/3} \)
\( = \frac{1000}{10} = 100 \)

Tangent line: \( y = 150,000 + 100 (L - 1000) \)

Note: If \( L \) is near 1000, then
\( Q(L) \approx 150,000 + 100 (L - 1000) \)
So the change in $Q$, from $L=1000$ to $L=975$ is approximately:

$$\Delta Q = Q(975) - Q(1000)$$

$$\approx \frac{150,000 + 100(975-1000)}{150,000} - \frac{150,000}{Q} = Q(1000)$$

$$\approx -2500$$

So $Q$ decreases by about 2500.

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**Ex. 4**  (Worksheet 3.6/#3)

Find equation of tangent line to

$$xe^y = 2xy + y^3$$

at the point $\left(\frac{1}{e-2}, 1\right)$.

Solution:

Implicitly differentiate equation w.r.t. $x$.

$$xe^y = 2xy + y^3$$
\[ 1 \cdot e^y + xe^y \cdot \frac{dy}{dx} = 2x \frac{dy}{dx} + 2y + 3y^2 \frac{dy}{dx} \]

Now substitute \( x = \frac{1}{e-2} \) and \( y = 1 \). Then solve for \( \frac{dy}{dx} \).

\[ e^1 + \frac{1}{e-2} \cdot e \cdot \frac{dy}{dx} = \frac{2}{e-2} \frac{dy}{dx} + 2 + 3 \frac{dy}{dx} \]

\[
\left( \frac{e}{e-2} - \frac{2}{e-2} - 3 \right) \frac{dy}{dx} = 2 - e
\]

\[
\frac{1}{e-2 - 3} \frac{dy}{dx} = 2 - e \quad (1)
\]

\[
(-2) \frac{dy}{dx} = 2 - e \quad (2)
\]

\[
\frac{dy}{dx} = \frac{2 - e}{-2} = \frac{e - 2}{2}
\]

So our tangent line is:
$y - 1 = \frac{e-2}{2} \left(x - \frac{1}{e-2}\right)$

**Ex. 5**

Where is $f(x) = 2x^3 + 15x^2 + 10$ increasing?

**Solution:**

$f'(x) = 6x^2 + 30x = 6x(x + 5)$

Recall: if $f'(x) > 0$ on $(a, b)$, then $f(x)$ is increasing on $(a, b)$. So we have to solve the inequality $f'(x) > 0$. The cut points for our sign chart satisfy $f'(x) = 0$, so our cut points are $x = 0$ and $x = -5$. The shape of $f$ change sign of $f'$ test point.
\[ f'(x) = 6x(x+5) \]
\[ f'(-6) = \bigcirc \bigcirc = \bigcirc \]
\[ f'(-1) = \bigcirc \bigcirc = \bigcirc \]
\[ f'(1) = \bigcirc \bigcirc = \bigcirc \]

So \( f(x) \) is increasing on \((-\infty, -5] \) and \([0, \infty) \).

(Acceptable: \((-\infty, -5) \cup (0, \infty)\))

**Ex. 6** (Spring 2019)

A child flies a kite at a constant height of 30 feet and wind carries the kite horizontally at 5 ft/sec. At what rate must the child let out the string when the kite is a distance of 50 ft. from the child?

Solution:
Information:
\[
\frac{dx}{dt} = 5
\]
\[
\frac{dL}{dt} = \text{???}
\]
When \( L = 50 \),

Substituting information into relations gives:
\[
x^2 + 30^2 = L^2 \quad (1)
\]
\[
10x = 100 \frac{dL}{dt} \quad (2)
\]
From (1), we have \( x = 40 \). So from (2), we get
\[
\frac{dL}{dt} = 4 \text{ ft/sec}.
\]

Ex. 7
Let \( f(x) = \frac{3e^x + 4}{2 - e^x} \).
(a) Find all vertical asymptotes. Then at each V.A. find both one-sided limits.

(b) Find all horizontal asymptotes.

Solution:

(a) Put \( 2 - e^x = 0 \), whence \( x = \ln(2) \) is the only V.A.

\[
\lim_{x \to \ln(2)^-} \left( \frac{3e^x + 4}{2 - e^x} \right) = \frac{0}{0} = \pm \infty
\]

\[
\lim_{x \to \ln(2)^+} \left( \frac{3e^x + 4}{2 - e^x} \right) = \frac{0}{0} = -\infty
\]

D.S. of \( x = \ln(2) \) gives \( \frac{10}{0} \).

To analyze sign of denominator, we have two methods:

- Method 1: Sign chart

\[
\begin{array}{c}
\oplus \quad \ominus \\
\ln(2) \\
\end{array}
\]

sign of \( 2 - e^x \)
\[
\ln(1.9) \quad \ln(2.1) \quad \text{test point}
\]

\[
2 - e^{\ln(1.9)} = 2 - e^{\ln(2.1)} =
\]

\[
2 - 1.9 = 0.1 > 0 \quad 2 - 2.1 = -0.1 < 0
\]

- **Method 2**: inspection of graph:

\[
\begin{align*}
y &= e^x \\
y &= -e^x \\
y &= 2 - e^x
\end{align*}
\]

So \( 2 - e^x > 0 \) for \( x < \ln(2) \) and \( 2 - e^x < 0 \) for \( x > \ln(2) \).

\((6)\) \[
\lim_{x \to \infty} \left( \frac{3e^x + 4}{2 - e^x} \right) = \lim_{x \to \infty} \left( \frac{3e^x}{-e^x} \right) = -3
\]

\(\frac{\infty}{-\infty}\), so use L'Hôpital's Rule

\[
\lim_{x \to -\infty} \left( \frac{3e^x + 4}{2 - e^x} \right) = \frac{0 + 4}{2 - 0} = 2
\]

Recall: \( \lim_{x \to -\infty} (e^x) = 0 \)
So the horizontal asymptotes are \( y = -3 \) and \( y = 2 \).

**Ex. 8** (Spring 2019)

According to post office regulations, the sum of the length and girth of a parcel cannot exceed 90 inches. What are the dimensions of the parcel with the largest volume if the parcel is rectangular with a square base?

**Solution:**

Let \( x \) be the width of the square base and \( y \) the length of the parcel.
**Objective:** \( V(x, y) = x^2 y \)

**Constraint:** \( 4x + \frac{y}{girth} = 90 \)

Solve for \( y \) in terms of \( x \)

\[ y = 90 - 4x \]

Substitute into objective.

\[ V(x, 90 - 4x) = x^2 (90 - 4x) \]

**Goal:** Find the value of \( x \) which gives absolute maximum value of

\[ f(x) = x^2 (90 - 4x) \]
on the interval \([0, \frac{9}{4}] = [0, 22.5]\).

Find critical #s:

- \(f'(x) = 0\) gives: none
- \(f'(x) = 0\):
  \[
  f(x) = 90x^2 - 4x^3
  \]
  \[
  f'(x) = 180x - 12x^2
  \]
  \[
  f'(x) = 12x(15-x) = 0
  \]
  \[
  x = 0 \quad \text{or} \quad x = 15
  \]

Now we verify that \(x = 15\) does give max value of \(f\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f(x) = x^2(90 - 4x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>22.5</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>225 - 30 &gt; 0</td>
</tr>
</tbody>
</table>

So the max value occurs when \(x = 15\).

So the dimensions of the box are
Ex. 9

The price per unit is

\[ p(x) = 180 - 3x - 2x^2 \]

if \( x \) units are being produced. Use marginal analysis to estimate the revenue derived from the 5th unit.

**Solution:**

Note that the total revenue is

\[ R(x) = xp(x) = 180x - 3x^2 - 2x^3 \]

We want \( MR(4) = R(5) - R(4) \), but we use the standard estimate:

\[ MR(4) \approx R'(4) \]

Observe:

\[ R'(x) = 180 - 6x - 6x^2 \]

\[ = -6(x^2 + x - 30) \]
\[ = -6(x+6)(x-5) \]

So now we have:

\[ R'(4) = -6(10)(-1) = 60 \]

So the revenue from the 5th unit is about 60.

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Ex. 10\(^\text{th}\) (Fall 2018)

Find an equation of tangent line to

\[ \sin\left(\frac{\pi x}{y}\right) = x - 8y \]

at the point (8, 1).

Solution:

To find \( \frac{dy}{dx} \), we use implicit diff.

\[ \cos\left(\frac{\pi x}{y}\right) \cdot \left(\frac{y \cdot \pi - \pi x \cdot \frac{dy}{dx}}{y^2}\right) = 1 - 8 \frac{dy}{dx} \]

\[ \frac{d}{dx}\left(\frac{\pi x}{y}\right) \]

Now substitute \( x = 8 \) and \( y = 1 \), then
solve for \( dy/dx \).

\[
\cos(8\pi) \cdot \left( \frac{\pi - 8\pi \cdot \frac{dy}{dx}}{1} \right) = 1 - 8 \frac{dy}{dx}
\]

\[
\pi - 8\pi \frac{dy}{dx} = 1 - 8 \frac{dy}{dx}
\]

\[
\frac{dy}{dx} = \frac{1}{8}
\]

So our tangent line is:

\[
y - 1 = \frac{1}{8} (x - 8)
\]

(or \( y = x/8 \))

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**Ex. 11**

Suppose \( f(x) \) is a function whose derivative is

\[
f'(x) = \frac{x}{(x^2 + 64)^{5/2}}
\]

Find the \( x \)-coordinate of each
Inflection point, if any.

Solution:
To find inflection points, we find where \( f''(x) \) changes sign, so we first solve the equation \( f''(x) = 0 \).

\[
f'(x) = \frac{x}{(x^2 + 64)^{\frac{3}{2}}}
\]

\[
f''(x) = \frac{(x^2 + 64)^{\frac{5}{2}} \cdot 1 - x \cdot \frac{5}{2} (x^2 + 64)^{\frac{3}{2}} \cdot 2x}{(x^2 + 64)^{\frac{5}{2}}}
\]

\[
f''(x) = \frac{(x^2 + 64)^{\frac{5}{2}} - 5x^2 (x^2 + 64)^{\frac{3}{2}}}{(x^2 + 64)^{\frac{5}{2}}}
\]

\[
f''(x) = \frac{(x^2 + 64)^{\frac{3}{2}} (x^2 + 64)^{\frac{1}{2}} - 5x^2}{(x^2 + 64)^{\frac{5}{2}}}
\]

\[
f''(x) = \frac{64 - 4x^2}{(x^2 + 64)^{\frac{7}{2}}}
\]
\[ f''(x) = \frac{4 \left(16 - x^2\right)}{(x^2 + 64)^{7/2}} \]

So the solutions to \( f''(x) = 0 \) satisfy:

\[ 4 \left(16 - x^2\right) = 0 \]

\[ x = -4 \quad \text{or} \quad x = 4 \]

Now verify these \( x \)-values really do give inflection points.

\[ \begin{array}{cccccccc}
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
-5 & -4 & + & 4 & - & 0 & 5
\end{array} \]

Shape of \( f \)

Sign of \( f'' \)

Test point

\[ f''(-5) = \frac{4 \left(16 - (-5)^2\right)}{(-5)^2 + 64)^{7/2}} = 0 \]

\[ f''(0) = \frac{4 \left(16 - 0^2\right)}{(0)^2 + 64)^{7/2}} = 0 \]

\[ f''(5) = \frac{4 \left(16 - 5^2\right)}{5^2 + 64)^{7/2}} = 0 \]
Since there is a change in sign in $f''(x)$ at both $x = -4$ and $x = 4$, there is an inflection point at each of these $x$-values.

**Ex. 12** (Fall 2017)

A camera is 5 feet away from a straight wire along which a bead moves to the right at 6 ft/sec. The camera turns so it always faces the bead directly. How fast is the camera turning 2 seconds after the bead passes closest to the camera?
Solution:

Information:
\[
\frac{dx}{dt} = 6 \\
\frac{d\theta}{dt} = ???
\]

Relations
\[
\tan(\theta) = \frac{x}{5} \\
\sec^2(\theta) \frac{d\theta}{dt} = \frac{1}{5} \cdot \frac{dx}{dt}
\]

When \( t = 2 \)

Note that when \( t = 2 \), we have the following:

\[
\begin{align*}
5 & \quad \theta \quad 13 \\
5 & \quad \theta \quad 12 \\
(\theta = 2.6)
\end{align*}
\]

\[
\Rightarrow \sec(\theta) = \frac{\text{HYP}}{\text{ADJ}} = \frac{13}{5}
\]

So from our second relation above, we have the following:
\[
\sec(\theta)^2 \cdot \frac{d\theta}{dt} = \frac{1}{5} \cdot \frac{dx}{dt}
\]

\[
\left(\frac{13}{5}\right)^2 \cdot \frac{d\theta}{dt} = \frac{1}{5} \cdot (6)
\]

\[
\Rightarrow \quad \frac{d\theta}{dt} = \frac{30}{169} \text{ radians/sec.}
\]