Review for Exam #1:

Ex. 3

Find an equation of the line tangent to the curve

\[ x^3 + e^{xy} = 3y + 9 \]

at the point (2, 0).

**Solution:**

Use implicit differentiation to find \( y' \):

\[ x^3 + e^{xy} = 3y + 9 \]

\[ 3x^2 + e^{xy} \cdot (xy' + y \cdot y') = 3y' \]

\[ \text{derivative of } xy \]

Substitute \( x = 2 \) and \( y = 0 \), then solve for \( y' \):

\[ 12 + e^0 (2y' + 1 \cdot 0) = 3y' \]

\[ 12 + 2y' = 3y' \]

\[ y' = 12 \text{ @ (2,0) only} \]

Equation of tangent line:

\[ y - 0 = 12 (x - 2) \]

Point: (2, 0), Slope: 12
Bonus: What is the normal line?

\[ y - 0 = -\frac{1}{12} (x-2) \]

point: \((2, 0)\), slope: \(-\frac{1}{12}\)

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Ex. 2

(a) Given the function \(g(x)\), state the definition of \(g'(4)\).

(b) Let \(F(x) = \frac{1}{3x-5}\). Find \(F'(2)\) using the limit definition of derivative.

Solution:

(a) Note: "slope of tangent line at \(x=4\)"
"rate of change of \(g\) at \(x=4\)"
"slope of function at \(x=4\)"
ALL WRONG! These are interpretations!
\[ g'(4) = \lim_{h \to 0} \frac{g(4+h) - g(4)}{h} \quad \text{OR} \quad \]

\[ g'(4) = \lim_{x \to 4} \frac{g(x) - g(4)}{x - 4} \]

(b) By definition,

\[ F'(2) = \lim_{h \to 0} \frac{F(2+h) - F(2)}{h} \]

\[ F(x) = \frac{1}{3x - 5} \]

\[ = \lim_{h \to 0} \left( \frac{1}{3(2+h) - 5} - 1 \right) \]

\[ = \lim_{h \to 0} \left( \frac{1}{3h+1} - 1 \right) \cdot \frac{3h+1}{3h+1} \]

\[ = \lim_{h \to 0} \left( \frac{1 - (3h+1)}{h(3h+1)} \right) \]

\[ = \lim_{h \to 0} \left( \frac{-3h}{h(3h+1)} \right) = \lim_{h \to 0} \left( \frac{-3}{3h+1} \right) \]
\[ \lim_{h \to 0} \left( \frac{-3}{3h+1} \right) = \frac{-3}{0+1} = -3 \]

**Check:**

\[ F(x) = \frac{1}{3x - 5} = (3x - 5)^{-1} \]

\[ F'(x) = -1 (3x - 5)^{-2} \cdot 3 \]

Chain rule: \( \frac{d}{dx} (3x - 5) = 3 \)

\[ F'(2) = -1 (6 - 5)^{-2} \cdot 3 = -3 \]

**Ex. 3**

(a) \[ \lim_{x \to 0} \left( \frac{(2x + 9)^2 - 81}{x} \right) \]

(b) \[ \lim_{x \to 3^-} \left( \frac{|x - 3|}{x - 3} \right) \]

(c) \[ \lim_{x \to 1} \left( \frac{5 - \sqrt{32 - 7x}}{x - 1} \right) \]

**Solution:**
(a) \[ \lim_{{x \to 0}} \left( \frac{{(2x+9)^2 - 81}}{x} \right) \] D.S. of \( x = 0 \) gives \( \frac{0}{0} \)

\[ = \lim_{{x \to 0}} \left( \frac{{4x^2 + 81 + 2 \cdot 2x \cdot 9 - 81}}{x} \right) \]

\[ = \lim_{{x \to 0}} \left( \frac{{4x^2 + 36x}}{x} \right) = \lim_{{x \to 0}} \left( 4x + 36 \right) = 36 \]

(b) \[ \lim_{{x \to 3^-}} \frac{{|x-3|}}{x-3} = (c.f, 2.2 \& 3.1 \text{ notes}) \]

Q: Is inside of \( |x-3| \) negative or positive?

A: If \( x \to 3^- \), this means \( x \) is close to 3 and \( x < 3 \), or \( x-3 < 0 \).

So \( |x-3| = -(x-3) \).

\[ = \lim_{{x \to 3^-}} \frac{{-(x-3)}}{x-3} = \lim_{{x \to 3^-}} \left( -1 \right) = -1 \]

(c) \[ \lim_{{x \to 1}} \left( \frac{{5 - \sqrt{32-7x}}}{x-1} \right) \] D.S. of \( x = 1 \) gives \( \frac{0}{0} \)
\[
\lim_{x \to 1} \left( \frac{5 - \sqrt{32-7x}}{x - 1} \cdot \frac{5 + \sqrt{32-7x}}{5 + \sqrt{32-7x}} \right)
\]

\[
= \lim_{x \to 1} \left( \frac{(5)^2 - (\sqrt{32-7x})^2}{(x-1)(5 + \sqrt{32-7x})} \right)
\]

\[
= \lim_{x \to 1} \left( \frac{25 - (32-7x)}{(x-1)(5 + \sqrt{32-7x})} \right)
\]

\[
= \lim_{x \to 1} \left( \frac{7(x-1)}{(x-1)(5 + \sqrt{32-7x})} \right)
\]

\[
= \lim_{x \to 1} \left( \frac{7}{5 + \sqrt{32-7x}} \right) = \frac{7}{5 + \sqrt{25}} = \frac{7}{10}
\]

Ex. 4

Show that the equation

\[x^{2/3} = 2x^2 + 2x - 2\]

has at least one solution in the interval \([0, 1]\). Explain your answer.

\underline{Solution:}

Use Intermediate Value Theorem (IVT). Equivalently, we show the following
equation has a solution in \([0, 1]\).

\[2x^2 + 2x - 2 - x^{2/3} = 0\]

Put \(f(x) = 2x^2 + 2x - 2 - x^{2/3}\). We need to show \(f(x) = 0\) for some \(x\) in \([0, 1]\). Observe:

- \(f(0) = 0 + 0 - 2 - 0 = -2\)
- \(f(1) = 2 + 2 - 2 - 1 = 1\)
- \(f\) is continuous on \([0, 1]\).

(Why? The function \(f\) is a sum of power functions and domain of \(f\) is \((-\infty, \infty)\). So \(f\) is continuous everywhere.)

Since 0 is between -2 and 1, the equation “\(f(x) = 0\)” is satisfied for at least one \(x\)-value in \([0, 1]\) (by the IVT).
Ex. 5

\[ g(s) = \begin{cases} 
3 & , \ s = 1 \\
\sqrt{1 - s} & , \ s < 1 \\
\frac{s^2 - s}{s - 1} & , \ s > 1 
\end{cases} \]

Calculate \( \lim_{x \to 1} g(x) \).

Solution:

Since \( x = 1 \) is the transition point for \( g(x) \), we must examine the left and right limits separately.

1. \( \lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (\sqrt{1 - x}) = 0 \)
2. \( \lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} \left( \frac{x^2 - x}{x - 1} \right) = \lim_{x \to 1^+} \frac{x(x-1)}{x-1} = \lim_{x \to 1^+} (x) = 1 \)

So \( \lim_{x \to 1} g(x) \) does not exist.
What is the role of "g(1) = 3"?
No role! But it would be relevant for a question on continuity.

Ex. 6

\[
f(x) = \begin{cases} 
\frac{\sin(ax)}{x}, & x < 0 \\
2x + 3, & 0 \leq x < 1 \\
b, & x = 1 \\
\frac{x^2 - 1}{x - 1}, & 1 < x
\end{cases}
\]

(a) Find the value of a so \( f \) is continuous at \( x = 0 \).
(b) Find the value of b so \( f \) is continuous at \( x = 1 \).

**Solution:**
(a) If \( f \) is to be continuous at \( x = 0 \), the left limit, right limit, and function value must be equal.
\[ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left( \frac{\sin(ax)}{x} \right) = \lim_{x \to 0^-} \left( \frac{\sin(ax)}{ax} \cdot a \right) \]

\[ = \lim_{x \to 0^-} \left( \frac{\sin(ax)}{ax} \right) \cdot \lim_{x \to 0^-} (a) = a \]

\[ = \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 = a \]

\[ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2x + 3) = 3 \]

\[ f(0) = (2x + 3) \bigg|_{x=0} = 3 \]

So we must choose \( a = 3 \).

(b) If \( f \) is to be continuous at \( x=1 \), the left limit, right limit, and function value must be equal.

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x + 3) = 5 \]

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left( \frac{x^2-1}{x-1} \right) = \lim_{x \to 1^+} \frac{(x-1)(x+1)}{x-1} \]
\[
\lim_{{x \to 1^+}} (x+1) = 2
\]

- \( f(1) = b \) (not necessary \( 5 \neq 2 \))

Since \( 5 \neq 2 \), no such value of \( b \) exists.

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**Ex. 7**

(a) Where is \( f \) not continuous in \((-5, 5)\)?

(b) Where is \( f \) not differentiable in \((-5, 5)\)?

(c) Where is \( f'(x) = 0 \) in \((-5, 5)\)?

(d) Where is \( f'(x) < 0 \) in \((-5, 5)\)?
(a) \( x = -3, \ x = -1 \)
(b) \( x = 3, \ x = -3, \ x = -1 \)

sharp corner \( f \) is discontinuous there

(c) \( x = 1 \) and all \( x \) in \((-3, -1)\)
(d) \((-5, -3) \cup (-1, 1) \cup (3, 5)\)

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**Ex. 8**

Find tangent line to \( y = 2x^2 - 3x + 1 \)

at \( x = 1 \).

**Solution:**

\( f'(x) = 4x - 3 \)

\( f'(1) = 4 - 3 = 1 \) \( \text{ (slope) } \)

\( f(1) = 2 - 3 + 1 = 0 \) \( \text{ (point: (1,0)) } \)

Equation of tangent line:

\[ y - 0 = 1(x - 1) \]

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**Ex. 9**
Find all solutions to the equation:

\[
2 \ln(x) = \ln \left( \frac{x^5}{5-x} \right) - \ln \left( \frac{x^3}{2+x} \right)
\]

**Solution:**

Write each side as a single logarithm.

LHS: \(2 \ln(x) = \ln(x^2)\) for \(x > 0\)

RHS: \[
\ln \left( \frac{\frac{x^5}{5-x}}{\frac{x^3}{2+x}} \right)
\]

\[
= \ln \left( \frac{x^5}{5-x} \cdot \frac{2+x}{x^3} \right) = \ln \left( \frac{x^2 (2+x)}{5-x} \right)
\]

So our equation is then

\[
\ln (x^2) = \ln \left( \frac{x^2 (2+x)}{5-x} \right)
\]
\[
\ln(x^2) = \ln\left(\frac{x^2(2+x)}{5-x}\right)
\]

\[
x^2 = \frac{x^2(2+x)}{5-x}
\]

\[
x^2(5-x) = x^2(2+x)
\]

\[
x^2(5-x) - x^2(2+x) = 0
\]

\[
x^2[(5-x) - (2+x)] = 0
\]

\[
x^2[5-x-2-x] = 0
\]

\[
x^2(3-2x) = 0
\]

\[
x^2 = 0 \quad \text{OR} \quad 3-2x = 0
\]

\[
x = 0 \quad \text{OR} \quad x = \frac{3}{2}
\]

Do these solve the original equation?

\[
x = 0: \text{ No, since } \ln(0) \text{ is undefined}
\]

\[
x = \frac{3}{2}: \text{ yes, a solution!}
\]