Section 5.4: Fundamental Theorem of Calculus

**Theorem #1: Fundamental Theorem (Part 1)**

Suppose $f$ is continuous on $[a, b]$. If $F$ is an antiderivative of $f$ on $(a, b)$,
\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

Recall Notation:
- $\int_{a}^{b} f(x) \, dx$:
  - Integral of $f$
  - Area under graph
  - A number
- $\int f(x) \, dx$:
  - The most general antiderivative of $f$
  - Family of functions

Derivative:
- Slope
- Rate of change

Integral:
- Area under graph

Limit:
- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- $\int_{a}^{b} f(x) \, dx = \lim_{N \to \infty} R_N$
So FTC1 tells us that if we have an antiderivative of \( f \), then we can use it to find integrals of \( f \).

\[
\text{Q: } \text{Does a general function have an antiderivative?} \\
\text{A: } \text{No. (But continuous functions do.)}
\]

\[
\text{Q: } \text{If the antiderivative exists, how do you find a useful formula for it?} \\
\text{A: } \text{Very difficult. (Calculus II.)}
\]

**Theorem #2: Fundamental Theorem (part 2)**  
Suppose \( f \) is continuous on \([a, b]\). Let \( x \in [a, b] \) and define

\[
A(x) = \int_a^x f(t) \, dt
\]

Then \( A \) is an antiderivative of \( f \) on \((a, b)\). That is, \( A'(x) = f(x) \), or
\[
\frac{d}{dx} \left( \int_a^x f(t) \, dt \right) = f(x)
\]

**Note**: Not all continuous functions are differentiable (e.g., \(|x|, (x-3)^{2/3}\), etc.). But all cont. functions have antiderivatives.

*Special notation for FTC:

\[
g(x) \bigg|_a^b = g(b) - g(a)
\]

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**Ex. 1**

Calculate the following integrals.

(a) \[\int_1^3 x^3 \, dx\]

\[
\int_1^3 x^3 \, dx = \frac{x^4}{4} \bigg|_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = 20
\]
(b) \[ \int_{-\pi/4}^{\pi/4} \sec^2(\theta) \, d\theta \]

\[ \int_{-\pi/4}^{\pi/4} \sec(\theta) \, d\theta = \tan(\theta) \bigg|_{-\pi/4}^{\pi/4} = \tan(\frac{\pi}{4}) - \tan(-\frac{\pi}{4}) \]

\[ = 1 - (-1) = 2 \]

(c) \[ \int_{0}^{\pi} \sin(\theta) \, d\theta \]

\[ \int_{0}^{\pi} \sin(\theta) \, d\theta = -\cos(\theta) \bigg|_{0}^{\pi} = [-\cos(\pi)] - [-\cos(0)] \]

\[ = [-(-1)] - [-1] = 2 \]

(d) \[ \int_{0}^{2\pi} \sin(\theta) \, d\theta \]

The positive area is 2, and the negative area is -2.
\[ \int_0^{2\pi} \sin(\theta) \, d\theta = -\cos(\theta) \bigg|_0^{2\pi} = (-\cos 2\pi) - (-\cos 0) = (-1) - (-1) = 0 \]

\[(e) \int_1^2 \frac{1}{x} \, dx \]

\[ \int_1^2 \frac{1}{x} \, dx = \ln \left| x \right| \bigg|_1^2 = \ln(2) - \ln(1) = \ln(2) \]

\[(f) \int_0^2 \frac{1}{x+1} \, dx \]

What is an antiderivative of the function \( f(x) = \frac{1}{x+1} \)?

\[ \frac{d}{dx} \ln(x+1) = \frac{1}{x+1} \cdot 1 = \frac{1}{x+1} \] √
Since coefficient of $x$ is 1, chain rule does not make finding the antiderivative any more difficult.

\[ \int_0^2 \frac{1}{x+1} \, dx = \ln(x+1) \bigg|_0^2 = \ln(3) - \ln(1) = \ln(3) \]

\[(g) \int_0^2 \frac{1}{2x+1} \, dx \]

What is an antiderivative of the function $f(x) = \frac{1}{2x+1}$? Not quite.

\[ \frac{d}{dx} \left( \ln(2x+1) \right) = \frac{1}{2x+1} \cdot 2 \]

Our antiderivative is off by a constant factor. So how do we fix this? Divide our candidate by that factor.
\[ \frac{d}{dx} \left( \frac{1}{2} \ln(2x+1) \right) = \frac{1}{2} \cdot \frac{1}{2x+1} \cdot 2 = \frac{1}{2x+1} \]

\[ \int_0^1 \frac{1}{2x+1} \, dx = \frac{1}{2} \ln(2x+1) \int_0^2 = \frac{1}{2} \ln(5) - \frac{1}{2} \ln(1) = \frac{1}{2} \ln(5) \]

(h) \[ \int_0^1 (3x+5)^{18} \, dx \]

Looks like \( u^{18} \)

\[ \frac{d}{dx} \left( \frac{(3x+5)^{19}}{19} \right) = \frac{19 \cdot (3x+5)^{18} \cdot 3}{19} \]

\[ \frac{d}{dx} \left( \frac{(3x+5)^{19}}{57} \right) = \frac{19 \cdot (3x+5)^{18} \cdot 3}{57} \]

\[ \int_0^1 (3x+5)^{18} \, dx = \frac{(3x+5)^{19}}{57} \bigg|_0^1 = \frac{8^{19}}{57} - \frac{5^{19}}{57} \]

Note: This method only works if coefficient of \( x \) is constant.
\[
\int \cos(x^2) \, dx \neq \sin(x^2) \cdot \frac{1}{2x} + C
\]

\[
\int \cos(x^2) \, dx \neq \sin(x^2) \cdot \frac{1}{x} + C
\]

(i) \[
\int_{\frac{8}{27}}^{1} \frac{4t^{4/3} - 10t^{1/3}}{t^2} \, dt \quad \text{(algebra)}
\]

= \[
\int_{\frac{8}{27}}^{1} \left( 4t^{-2/3} - 10t^{-5/3} \right) \, dt
\]

= \[
\left( 4 \cdot \frac{t^{1/3}}{1/3} - 10 \cdot \frac{t^{2/3}}{-2/3} \right) \bigg|_{\frac{8}{27}}^{1}
\]

= \[
\left( 12t^{1/3} + 15t^{-2/3} \right) \bigg|_{\frac{8}{27}}^{1}
\]

Note: \[
\left( \frac{8}{27} \right)^{1/3} = \frac{2}{3}, \quad \left( \frac{8}{27} \right)^{-2/3} = \frac{9}{4}
\]

= \[
\left[ 12 \cdot 1 + 15 \cdot 1 \right] - \left[ 12 \cdot \frac{2}{3} + 15 \cdot \frac{9}{4} \right] = -\frac{59}{4}
\]

Ex. 2
Suppose $x \geq -1$. Let
\[ G(x) = \int_1^x \sqrt{t^3 + 1} \, dt \]

(a) Calculate $G(1)$.
(b) Calculate $G'(2)$.

\[ \text{Solution:} \]

(a) $G(1) = \int_1^1 \sqrt{t^3 + 1} \, dt = 0$

(b) Recall FTC 2:

\[ \frac{d}{dx} \left( \int_a^x f(t) \, dt \right) = f(x) \]

Upper limit is $x$. Replace dummy variable with $x$

Lower limit is a constant
So we have

\[ G'(x) = \frac{d}{dx} \left( \int_{1}^{x} \sqrt{t^3 + 1} \, dt \right) = \sqrt{x^3 + 1} \]

So we have \( G'(2) = \sqrt{8+1} = 3 \).

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**Ex. 3**

Let \( f(x) = \begin{cases} 
  e^x & \text{if } x < 0 \\
  1-x^2 & \text{if } x \geq 0 
\end{cases} \)

(a) Can we use FTC1 to calculate the integral \( \int_{-1}^{1} f(x) \, dx \)?

(b) Calculate \( \int_{-1}^{1} f(x) \, dx \).

**Solution:**

(a) Equivalently, we want to determine whether \( f(x) \) is continuous on \([-1, 1]\). Since each piece is continuous on \((-\infty, \infty)\), we only need to check whether \( f \) is
continuous at \( x = 0 \). We have:

\[
\lim_{{x \to 0^-}} f(x) = \lim_{{x \to 0^-}} (e^x) = e^0 = 1
\]

\[
\lim_{{x \to 0^+}} f(x) = \lim_{{x \to 0^+}} (1-x^2) = 1-0 = 1
\]

\[
f(0) = (1-x^2)|_{{x=0}} = 1
\]

Since \( \lim_{{x \to 0^-}} f(x) = \lim_{{x \to 0^+}} f(x) = f(0) \), \( f \) is continuous at \( x = 0 \).

(b)

\[
\int_{-1}^{1} f(x) \, dx = \int_{-1}^{0} f(x) \, dx + \int_{0}^{1} f(x) \, dx
\]

\[
\int_{-1}^{0} f(x) \, dx = \int_{-1}^{0} e^x \, dx = e^x \bigg|_{{-1}}^{0}
\]
\[ e^0 - e^{-1} = 1 - \frac{1}{e} \]

\[ \int_0^1 f(x) \, dx = \int_0^1 (1 - x^2) \, dx = \left( x - \frac{x^3}{3} \right) \bigg|_0^1 \]

\[ = \left( 1 - \frac{1}{3} \right) - (0 - 0) = \frac{2}{3} \]

So the total area is

\[ \int_0^1 f(x) \, dx = 1 - \frac{1}{e} + \frac{2}{3} = \frac{5}{3} - \frac{1}{e} \]