Maximizing Profit

Profit: \( \pi(x) = R(x) - C(x) \)

To maximize profit, we find critical points of \( \pi(x) \):

\( \pi'(x) = 0 \)

\[ R'(x) - C'(x) = 0 \]

\[ R'(x) = C'(x) \]

Recall: From Section 3.8,

\[ MR(x) = R(x+1) - R(x) \approx R'(x) \]

\[ MR(x) = MC(x) \]

Technically, this is an approximation, but it is used pervasively in economics.

**Ex. 1**

Suppose total cost and market price are given by
Find optimal level of production.

Solution:

"Optimal level of production" always means "value of \( x \) for which profit is a maximum. (Unless explicitly otherwise stated.)

Revenue: \( R(x) = x \cdot p(x) = \frac{1}{5} (62x - x^2) \)

Cost: \( C(x) = \frac{2}{5} x^2 + 4x + 44 \)

To maximize profit, solve "\( MR = MC \)"

\[
MR = MC \\
\left( R'(x) = C'(x) \right)
\]

\[
\frac{1}{5} (62 - 2x) = \frac{4}{5} x + 4
\]

\[
62 - 2x = 4x + 20
\]
Now verify that \( x = 7 \) really does give max profit.... but wait!

Economic theory guarantees that revenue and cost satisfy certain conditions so that solving "\( MR = MC \)" is sufficient for finding max profit.

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**Ex. 2**

All units in a 100-unit apartment complex are rented if the rent is $900/month. One unit becomes vacant for each $10 increase in rent, and each occupied unit costs $80/month in maintenance. What is the optimal rent per unit?
Solution:
We want to use “MR=MC” to find max profit, but we need to find revenue and cost first.

Let $x$ be the # of rented units and let $p(x)$ be the monthly rent per unit. We need to determine $p$ in terms of $x$. What relationship exists between $p$ and $x$?

- If $\Delta p = 10$, then $\Delta x = -1$
- If $\Delta p = 20$, then $\Delta x = -2$
- If $\Delta p = 30$, then $\Delta x = -3$

So note that $\Delta p$ and $\Delta x$ are always in the same proportion, independent of current values of $x$ and $p$.

The phrase “for each” tells us that
p(x) is a linear function of x.

\[ \frac{\Delta p}{\Delta x} \text{ not constant} \]

\[ \frac{\Delta p}{\Delta x} \text{ constant} \]

Point on graph of \( y = p(x) \): \( (100, 900) \)

\( \rightarrow \) all 100 units are rented if rent is $900.

So \( p = 900 \) if \( x = 100 \).

Slope of \( y = p(x) \): \( m = \frac{\Delta p}{\Delta x} = \frac{10}{-1} = -10 \)

\( \rightarrow \) if price increases by $(10, price decreases by $(10).
occupied units decreases by 1.
So if $\Delta p = 10$, then $\Delta x = -1$.

**Equation for $p(x)$:**

$$p(x) = 900 - 10(x - 100)$$

$$p(x) = 1900 - 10x$$

**Revenue:** $R(x) = xp(x) = 1900x - 10x^2$

**Cost:** $C(x) = 80x$

To maximize profit, we solve “$MR = MC$”.

$$MR = MC$$

$$1900 - 20x = 80$$

$$1820 = 20x$$

$$91 = x$$  \(\text{optimal # of rented units}\)

So the optimal rent is $p(91) = 990$ \(\$/\text{month}\)
No verification necessary for “MR=MC” problems.

Ex. 3

Given that the total cost of producing $x$ widgets is

$$C(x) = 3x^2 + x + 48,$$

find the level of production which minimizes the average cost per unit.

Solution:

By definition, the average cost per unit is

$$AC(x) = \frac{C(x)}{x}$$

So for our problem, we want to find the value of $x$ which minimizes
\[ AC(x) = \frac{3x^2 + x + 48}{x} = 3x + 1 + \frac{48}{x} \]

on the interval \( 0 < x < \infty \).

Critical \#'s of \( AC(x) \):  
- \( AC'(x) \) dne: none  
- \( AC'(x) = 0 \):  
\[ AC'(x) = 3 - \frac{48}{x^2} = 0 \]

\[ x^2 = 16 \]
\[ x = -4 \quad \text{or} \quad x = 4 \]

Cannot produce negative units

Since this is not a “MR=MC” type problem, we must verify that \( x = 4 \) gives the minimum average cost.

Observe that
\[ AC''(x) = \frac{96}{x^3} \]

and \[ AC''(x) > 0 \] for all \( x > 0 \). So this means that the graph of \( y = AC(x) \) is concave up on \((0, \infty)\).

\[ \text{So } x = 4 \text{ gives the global minimum value of } AC(x) \text{ for } x > 0. \]

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**Ex. 4**

The manufacturing cost of producing \( x \) widgets is

\[ C(x) = \frac{1}{10}x + 6 \]

\[ \text{fixed cost or sunk cost } C(0) = 6 \]

and the market price is

\[ p(x) = 70 - x \]
In addition, the manufacturer must pay a government tax of $0.10 per unit produced. What is the optimal level of production?

**Solution:**

We will use "MR = MC" to maximize profit.

**Revenue:** \( R(x) = xp(x) = \frac{70x - x^2}{30 + x} \)

**Cost:** \( \tilde{C}(x) = C(x) + \frac{1}{10}x \)

Now to maximize profit, we solve
"MR = M\tilde{C}".

\[
MR = M\tilde{C} \\
\frac{(30+x)(70-2x)-(70x-x^2)(1)}{(30+x)^2} = \frac{1}{5}
\]

\[
-x^2 - 60x + 2100 = \frac{1}{5}(x+30)^2
\]

\[
-x^2 - 60x + 2100 = \frac{1}{5}x^2 + 12x + 180
\]

\[
0 = \frac{6}{5}x^2 + 72x - 1920
\]

\[
0 = \frac{6}{5}(x+80)(x-20)
\]

\[
x = -80 \quad \text{or} \quad x = 20
\]

cannot produce negative units

No need to verify we have max
Ex. 5

Store sells skateboards at $40 per board and skaters buy 45 boards per month. Owner estimates that for each $1 increase in price, 3 fewer boards will sell. Each board costs the owner $29. What is the optimal price?

Solution:

We will use “MR = MC” to find max profit. First we find revenue and cost. Let x be the # of boards sold and let p(x) be the price per board.

The phrase “for each” tells us that Δp is constant, i.e., p(x) is a linear function of x.
linear function of $x$.

Point on graph of $y = p(x)$: $(45, 40)$

$\rightarrow$ if $x = 45$, then $p = 40$

Slope of $y = p(x)$: $m = \frac{\Delta p}{\Delta x} = \frac{1}{-3} = -\frac{1}{3}$

$\rightarrow$ if $\Delta p = 1$, then $\Delta x = -3$

equation for $p(x)$:

$$p(x) = 40 - \frac{1}{3} (x-45)$$

$$p(x) = 55 - \frac{1}{3} x$$

Revenue: $R(x) = x p(x) = 55x - \frac{1}{3} x^2$

Cost: $C(x) = 29x$

Now to maximize profit, we solve “MR = MC”.
MR = MC

\[ 55 - \frac{2}{3} x = 29 \]

\[ 26 = \frac{2}{3} x \]

\[ 39 = x \quad \leftarrow \text{optimal \# of boards sold} \]

So the optimal price is \( p(39) = 42 \) ($/board)

No verification necessary for “MR = MC” problems.