Section 4.6: Optimization

Ex. 1

The difference of two numbers is 10. Find their minimum product.

Solution:

Let \( x \) and \( y \) be the two numbers. We want to minimize the function

\[
p(x, y) = xy
\]

Objective function

Problem: This is a function of more than one variable (uh-oh!)

However, \( x \) and \( y \) are not independent of each other. Instead they satisfy the constraint equation

\[
x - y = 10
\]

We use the constraint to write the objective in terms of one variable.
\[ x = y + 10 \]

\[ \Rightarrow \text{Substitute into objective function} \]

Our objective function is now

\[ p(y + 10, y) = (y + 10)y = y^2 + 10y \]

**Goal:** Find the absolute minimum value of the function

\[ f(y) = y^2 + 10y \]

on the interval \((-\infty, \infty)\).

\[ \Rightarrow \text{In Math III/115, this is a valid problem up until this point.} \]

Find the critical #'s:

- \( f'(y) \) does: none
- \( f'(y) = 0 \):

\[ f'(y) = 2y + 10 = 0 \]

\[ \Rightarrow y = -5 \]

Since \((-\infty, \infty)\) is not bounded, we
Cannot use the method of Section 4.1 to determine the absolute minimum. Worse, the minimum may not even exist. So we will use methods of Sections 4.3 and 4.4.

What does $f'$ tell us? Construct a sign chart for $f'$.

\[ f'(y) = 2y + 10 \]

- $f'(-6) = -2 < 0$
- $f'(0) = 10 > 0$

Since $f$ is decreasing on $(-\infty, -5]$ and increasing on $[-5, \infty)$, absolute min of $f$ occurs at $y = -5$. So the minimum product is $f(-5) = -25$. 
A cylindrical tank has volume \(2000\pi\, \text{m}^3\). Find the dimensions of the tank with the smallest possible surface area.

**Hint:** \[ A = 2\pi r^2 + 2\pi rh \]

\[ V = \pi r^2 h \]

**Solution:**

We want to minimize the function

\[ A(r, h) = 2\pi r^2 + 2\pi rh \]

subject to the constraint equation

\[ 2000\pi = \pi r^2 h \]

The only role of this equation is
to let us write \( A(r,h) \) in terms of one variable only. We never differentiate the constraint equation.

Solving for \( h \) in terms of \( r \):

\[
h = \frac{2000}{r^2}
\]

Substituting into the objective gives

\[
A(r, \frac{2000}{r^2}) = 2\pi r^2 + 2\pi r \left( \frac{2000}{r^2} \right)
\]

\[
= 2\pi \left( r^2 + \frac{2000}{r} \right)
\]

**Goal:** Find the value of \( r \) that gives absolute minimum value of the function

\[
f(r) = 2\pi \left( r^2 + \frac{2000}{r} \right)
\]

on the interval \((0, \infty)\)

Find the critical #\(^{s}\):

- \( f'(r) \) does not exist (c.f., Section 4.2)
Now determine the absolute minimum.

**Method 1:** using first derivative

\[ f'(r) = 2\pi \left( 2r - \frac{2000}{r^2} \right) = 0 \]

\[ 2r^3 = 2000 \]

\[ r^3 = 1000 \]

\[ r = 10 \]

Since \( f \) is decreasing on \((0, 10]\) and increasing on \([10, \infty)\), absolute min of \( f \) occurs at \( r = 10 \).
Method 2: using second derivative

\[ f''(r) = 2\pi \left( 2r - \frac{2000}{r^2} \right) \]

\[ f''(r) = 2\pi \left( 2 + \frac{4000}{r^3} \right) \]

\[ > 0 \quad > 0 \quad > 0 \quad \text{if} \quad r > 0 \]

So \( f''(r) > 0 \) for all \( r > 0 \). So \( f \) is concave up on \((0, \infty)\). Since \( r = 10 \) is the only critical #, absolute min of \( f \) occurs at \( r = 10 \).

Final Answer: \( r = 10, \quad h = \frac{2000}{10^2} = 20 \)

Ex. 3

A rectangle has its lower left vertex at the origin and its upper right vertex on the graph of \( y = \frac{4-x}{2+x} \). Find the largest possible area of such a rectangle.
Solution:

Let \( w \) and \( h \) be the width and height of the rectangle, respectively. We want to maximize the function

\[
A(w, h) = wh
\]

subject to the constraint equation

\[
h = \frac{4-w}{2+w}
\]
Substituting into the objective gives

\[ A(w, \frac{4-w}{2+w}) = w \cdot \frac{4-w}{2+w} = \frac{4w-w^2}{2+w} \]

**Goal:** Find the absolute maximum area of the function

\[ f(w) = \frac{4w-w^2}{2+w} \]

on the interval \([0, 4]\).

Find the critical #s:

- \( f'(w) \) due: none  \( \text{(c.f. Section 4.2)} \)
- \( f''(w) = 0 \):

\[
\begin{align*}
    f'(w) &= \frac{(2+w)(4-2w)-(4w-w^2)}{(2+w)^2} \\
    f'(w) &= \frac{8-4w-w^2}{(2+w)^2} = 0 \\
    8-4w-w^2 &= 0 \\
    \quad \quad \quad \quad \quad \quad \quad \quad \quad w = -2 - 2\sqrt{3} \quad \text{or} \quad w = -2 + 2\sqrt{3}
\end{align*}
\]
Now determine the absolute max of $f$. Since $[0, 4]$ is closed and bounded, we can use the method of Section 4.1.

$$f(w) = \frac{w(4-w)}{2+w}$$

<table>
<thead>
<tr>
<th>w-values</th>
<th>y-values</th>
</tr>
</thead>
<tbody>
<tr>
<td>endpt 0</td>
<td>0</td>
</tr>
<tr>
<td>endpt 4</td>
<td>0</td>
</tr>
<tr>
<td>critical # $-2+2\sqrt{3} \approx 1.4$</td>
<td>$\frac{(1.4)(2.6)}{3.4} &gt; 0$</td>
</tr>
</tbody>
</table>

So the absolute maximum of $f$ is

$$f(-2+2\sqrt{3}) = \frac{(-2+2\sqrt{3})(6-2\sqrt{3})}{2\sqrt{3}} = 8-4\sqrt{3}.$$
times its width.

(a) Find $f(w)$, the volume of the box in terms of the width $w$. What is the domain of $f$ in the context of this problem?

(b) Find the dimensions of such a box with the largest possible volume.

Solution:

(a) Let $l$, $w$, $h$ be the length, width and height of the box, respectively. We want to maximize the function

$$V(l,w,h) = lwh$$

subject to the constraint equations

$$l = 3w \quad (1)$$

$$450 = 2lw + 2lh + 2wh \quad (2)$$
Put (1) into (2):

(2)* \[450 = 6w^2 + 6wh + 2wh\]

Now solve for \(h\) in terms of \(w\).

\[450 = 6w^2 + 8wh\]

\[\implies h = \frac{450 - 6w^2}{8w} = \frac{225 - 3w^2}{4w}\]

Now rewrite objective in terms of \(w\).

\[V(3w, w, \frac{225 - 3w^2}{4w}) = 3w \cdot w \cdot \frac{225 - 3w^2}{4w}\]

\[f(w) = \frac{3}{4} (225w - 3w^3)\]

What are the allowed values of \(w\)?

Since all length are \(> 0\), we must have
The domain of \( f(w) \) is \((0, \sqrt{75}]\).

**Goal:** Find the value of \( w \) that gives the absolute maximum value of the function
\[
f(w) = \frac{3}{4} (225w - 3w^3)
\]
on the interval \((0, \sqrt{75}]\).

(b) Find the critical #s.

- \( f'(w) \) dne: none
- \( f'(w) = 0 \):

\[
f'(w) = \frac{3}{4} (225 - 9w^2) = 0
\]
\[
9w^2 = 225 \Rightarrow w^2 = 25 \Rightarrow w = 5
\]
Determine absolute maximum of $f$. Use the second derivative.

$$f''(w) = \frac{3}{4} (-18w) = -\frac{27}{2}w$$

Observe that $f''(w) < 0$ for $w$ in $(0, \sqrt{75}]$. So $f$ is concave down on $(0, \sqrt{75}]$. Since $w = 5$ is the only critical #, $w = 5$ gives absolute max.

So the dimensions of the box with the largest volume are $l = 15, w = 5$, and $h = 7.5$.

Ex. 5

Find the equation of the line through $P = (12, 4)$ such that the triangle bounded by the line and coordinate axes has minimum area.

Solution:
Any line that passes through \( P \) must have an equation of the form

\[
y = 4 + m(x - 12)
\]

where \( m \) is an unknown slope. So our variable will be the slope \( m \).

Find width and height of triangle in terms of \( m \):

**width:** \((w,0)\) lies on line, so...

\[
0 = 4 + m(w - 12) \implies w = -\frac{4}{m} + 12
\]

formula for \( w \) in terms of \( m \)
height: \((0, h)\) lies on line, so...
\[ h = 4 + m(0 - 12) \Rightarrow h = 4 - 12m \]

formal formula for \(h\) in terms of \(m\)

So the area of the triangle is

\[ f(m) = \frac{1}{2} \cdot \left( -\frac{4}{m} + 12 \right) (4 - 12m) \]

area of \(\Delta = \frac{1}{2}wh\)

Note: The line must have a negative slope!

Goal: Find value of \(m\) that gives the minimum value of function

\[ f(m) = 48 - \frac{8}{m} - 72m \]

on the interval \((-\infty, 0)\).

Find critical #'s:

- \(f'(m)\) dne: nowhere
\[ f'(m) = 0:\]
\[ f'(m) = \frac{8}{m^2} - 72 = 0 \]
\[ m^2 = \frac{1}{9} \]
\[ m = -\frac{1}{3} \quad \text{or} \quad m = \frac{1}{3} \]

OK: in \((\infty, 0)\)

not in \((-\infty, 0)\)

Now determine the absolute min of \(f\).
Use the second derivative!

\[ f''(m) = -\frac{16}{m^3} \]

Observe that \(f''(m) > 0\) if \(m < 0\), i.e.,
if \(m\) is in \((-\infty, 0)\). So \(f\) is concave up on \((-\infty, 0)\). Since \(m = -\frac{1}{3}\) is the only critical point, \(f\) must have an absolute minimum at \(m = -\frac{1}{3}\). The equation of the line is thus
\[ y = 4 - \frac{1}{3} (x-12) \]
A farmer wants to build a rectangular enclosure with 4 pens of equal size, all of which border a river (the side with the river needs no fencing). Each pen must have an area of 80 ft$^2$. Find the minimum amount of fencing needed to build the enclosure.

Solution:
Let \( x \) and \( y \) be the length and width of one pen, as indicated above. The objective function is:

\[
p(x, y) = 4x + 5y
\]

The constraint equation is:

\[
x \cdot y = 80
\]

Solving for \( y \) in constraint gives

\[
y = \frac{80}{x}
\]

Substituting into the objective gives

\[
p(x, \frac{80}{x}) = 4x + 5 \left( \frac{80}{x} \right) = 4x + \frac{400}{x}
\]

**Goal:** Find absolute minimum value of the function

\[
f(x) = 4x + \frac{400}{x}
\]
on the interval \((0, \infty)\).
Find critical #’s:

• \( f'(x) \) due: none

• \( f'(x) = 0 \):

\[
f'(x) = 4 - \frac{400}{x^2} = 0
\]

\[x^2 = 100\]

\[x = -10 \text{ or } x = 10\]

Not in \((0, \infty)\)

Now observe that

\[
f''(x) = \frac{800}{x^3}
\]

So \( f''(x) > 0 \) for all \( x > 0 \), whence \( f \) is concave up on \((0, \infty)\). So \( x = 10 \) gives min value of \( f \) on \((0, \infty)\).

**Final answer:** minimum fencing is \( f(10) = 80 \text{ ft.} \)