Section 4.3: Shapes of Graphs

* Be sure to review supplementary notes for Section 4.3: “Conceptual Background for Shapes of Graphs”

Ex. 1

Graph \( f(x) = x^3 - 12x^2 \) on \([-1, 9]\).

Solution:

\[
\begin{align*}
  f(x) &= x^3 - 12x^2 = x^2(x-12) \\
  f'(x) &= 3x^2 - 24x = 3x(x-8) \\
  f''(x) &= 6x - 24 = 6(x-4)
\end{align*}
\]

1. Information from \( f(x) \):
   (Polynomials have no asymptotes)

2. Information from \( f'(x) \):
   First-order critical #’s:
   (Recall \( f'(x) = 3x(x-8) \))
   - \( f'(x) \) dne: none
   - \( f'(x) = 0 \): \( x=0, x=8 \)

Now construct sign chart for \( f'(x) \):
shape of \( f \)  
\[
\text{sign of } f' \text{ test point}
\]

\[
f'(x) = 3x(x-8)
\]

\[
f'(-1) = -15 = -
\]

\[
f'(1) = 15 = -
\]

\[
f'(9) = 18 = +
\]

\( f \) is decreasing on \([0, 8]\)  
acceptable: \((0, 8)\)

\( f \) is increasing on \((-\infty, 0]\), \([8, \infty)\)  
comma-separated list!  
acceptable: \((-\infty, 0) \cup (8, \infty)\)

local min @ \( x = 8 \)

local max @ \( x = 0 \)

\( \square \) information from \( f''(x) \):
Second-order critical #\((s)\):
(Recall \( f''(x) = 6(x-4) \))

\( f''(x) \) due: none
Now construct sign chart for $f''(x)$:

$$f''(x) = 6(x-4)$$

- $f''(0) = -6$: circle
- $f''(4) = 0$: circle
- $f''(5) = 6$: circle

- $f(x)$ is concave down on $(-\infty, 4]$ acceptable: $(-\infty, 4)$
- $f(x)$ is concave up on $[4, \infty)$ acceptable: $(4, \infty)$
- Inflection points at $x = 4$

$\text{Graph}$ $y = f(x)$:
Recall $f(x) = x^2(x-12)$ on $[-1, 9]$

$\text{Important Points on Graph}$:

<table>
<thead>
<tr>
<th>x-value</th>
<th>y-value</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-13</td>
<td>endpoint</td>
</tr>
</tbody>
</table>
Summary of Info from \( f, f', \) and \( f'' \):

- Local max at \((0, 0)\)
- Inflection point at \((4, -128)\)
- Local min at \((8, -256)\)

<table>
<thead>
<tr>
<th>Inc/dec</th>
<th>Concave up/down</th>
</tr>
</thead>
<tbody>
<tr>
<td>inc.</td>
<td>concave down</td>
</tr>
<tr>
<td>dec.</td>
<td>concave up</td>
</tr>
</tbody>
</table>

Graph showing:
- Local max at \((0, 0)\)
- Inflection point at \((4, -128)\)
- Local min at \((8, -256)\)
(Make sure to label important points. Graph does not have to be to scale.)

Ex. 2

Graph \( f(x) = x(x-2)^3 \) on \([-1, 3]\).

Solution:
\[
\begin{align*}
f(x) &= x(x-2)^3 \\
f'(x) &= 1 \cdot (x-2)^3 + x \cdot 3(x-2)^2 \cdot 1 \\
&= (x-2) + 3x \\
&= 2(2x-1) \\
f''(x) &= 2 \left[ 2 \cdot (x-2)^2 + (2x-1) \cdot 2(x-2) \cdot 1 \right] \\
&= 2(x-2)(2x-1) + 2(2x-1) \\
&= 12(x-2)(x-1)
\end{align*}
\]

(Don't expand. Just use product rule.)

1. Information from \( f(x) \):
   (Polynomials have no asymptotes)
2. Information from \( f'(x) \):
   First-order critical #'s:
   (Recall \( f'(x) = 2(2x-1)(x-2)^2 \))
   \begin{align*}
   &\cdot f'(x) \text{ dne: none} \\
   &\cdot f'(x) = 0: \ x = \frac{1}{2}, \ x = 2
   \end{align*}
Now construct sign chart for $f'(x)$:

Shape of $f$:

<table>
<thead>
<tr>
<th></th>
<th>$-\infty$</th>
<th>$1/2$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign</td>
<td>$-$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Test point

$f'(x) = 2(2x-1)(x-2)^2$

$f'(0) = +$ $-$ $+$ $-$ $-$ $-$ 

$f'(1) = +$ $+$ $+$ $+$ $+$ $+$ 

$f'(3) = +$ $+$ $+$ $+$ $+$ $+$ 

$f$ is decreasing on $(-\infty, \frac{1}{2}]$

$f$ is increasing on $[\frac{1}{2}, \infty)$

Not two separate intervals!

Local min @ $x = \frac{1}{2}$

Local max @ nowhere

Information from $f''(x)$:

Second-order critical #s:

(Recall $f''(x) = 12(x-2)(x-1)$)

- $f''(x)$ dne: none
- $f''(x) = 0$: $x = 1, x = 2$
Now construct sign chart for $f''(x)$:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f''(x)$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

$f''(0) = + \quad - \quad - \quad + = +$
$f''(1.5) = + \quad - \quad + \quad + = -$
$f''(3) = + \quad + \quad + \quad + = +$

$f$ is concave down on $[1, 2]$
$f$ is concave up on $(-\infty, 1], [2, \infty)$
Inflection point(s) @ $x = 1, x = 2$

Graph $y = f(x)$:
Recall $f(x) = x(x-2)^3$ on $[-1, 3]$

Important Points on Graph:

<table>
<thead>
<tr>
<th>x-value</th>
<th>y-value</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>27</td>
<td>endpoint</td>
</tr>
<tr>
<td>1/2</td>
<td>-27/16</td>
<td>local min</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>inflection point</td>
</tr>
</tbody>
</table>
Summary of Info from $f$, $f'$, and $f''$:

<table>
<thead>
<tr>
<th>Inc/Dec</th>
<th>Concave Up/Down</th>
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</thead>
<tbody>
<tr>
<td>dec.</td>
<td>concave up</td>
</tr>
<tr>
<td>increasing</td>
<td>concave down</td>
</tr>
<tr>
<td>concave up</td>
<td>concave up</td>
</tr>
</tbody>
</table>

Graph shows:
- Inflection point $\left(2, 0\right)$
- Inflection point $\left(1, -1\right)$
- Local min $\left(\frac{1}{2}, -\frac{27}{16}\right)$

(Make sure to label important points. Graph does not have to be to scale.)
Graph: \( f(x) = \frac{x}{x^2 - 4} \).

Solution:

\[
f'(x) = -\frac{(x^2 + 4)}{(x^2 - 4)^2}, \quad f''(x) = \frac{12x(x^2 + 12)}{(x^2 - 4)^3}
\]

1. Information from \( f(x) \):
   - Horizontal Asymptotes:
     \[
     \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x}{x^2 - 4} = \lim_{x \to \infty} \left( \frac{1}{x} \cdot \frac{1}{1 - \frac{4}{x^2}} \right) = 0 \cdot \frac{1}{1 - 0} = 0
     \]

     \[
     \lim_{x \to -\infty} f(x) = 0 \quad \text{(same as previous calculation)}
     \]

     The only HA is \( y = 0 \).

   - Vertical Asymptotes
     \( (\text{Look at where denominator is } 0). \)

     \[ x^2 - 4 = 0 \Rightarrow x = -2 \quad \text{or} \quad x = 2. \]
\[
\lim_{x \to -2^-} \frac{x}{x^2 - 4} = \frac{0}{\infty} = -\infty \quad \text{if } x \to -2^-, \ x^2 - 4 > 0
\]
\[
\lim_{x \to -2^+} \frac{x}{x^2 - 4} = \frac{0}{-\infty} = +\infty \quad \text{if } x \to -2^+, \ x^2 - 4 < 0
\]

D.S. of \( x = -2 \) gives \(-\frac{2}{0}\) for both.

So each limit is infinite. Positive or negative?

\[
\lim_{x \to 2^-} \frac{x}{x^2 - 4} = \frac{0}{\infty} = -\infty \quad \text{if } x \to 2^-, \ x^2 - 4 < 0
\]
\[
\lim_{x \to 2^+} \frac{x}{x^2 - 4} = \frac{0}{\infty} = +\infty \quad \text{if } x \to 2^+, \ x^2 - 4 > 0
\]

D.S. of \( x = 2 \) gives \(\frac{2}{0}\) for both.

So each limit is infinite. Positive or negative?

So \( x = -2 \) and \( x = 2 \) are UA's.

(2) Information from \( f'(x) \):

Recall: \( f'(x) = \frac{-\left(x^2 + 4\right)}{(x^2 - 4)^2} \)
First-order critical #'s
• \( f'(x) \) due: none \((x = \pm 2 \) not in domain of \( f \), so not critical #\)

• \( f'(x) = 0: \ - (x^2 + 4) = 0 \)
  no solution

New construct sign chart for \( f'(x) \):
(Critical #’s AND vertical asymptotes are cut points on sign chart.)

\[
f'(x) = \frac{-(x^2 + 4)}{(x^2 - 4)^2}
\]

\( f'(-3) = \frac{\text{--} \text{+}}{\text{+}} = 0 \)

\( f'(0) = \frac{\text{-} \text{+}}{\text{+}} = 0 \)

\( f'(3) = \frac{\text{-} \text{+}}{\text{+}} = 0 \)

\( f \) is decreasing on \((-\infty, -2), (-2, 2), (2, \infty)\)
f is increasing on no interval
f has local min @ no x-value
f has local max @ no x-value

3) Information from f"(x):

Recall: f"(x) = \frac{12x \left(x^2 + 12\right)}{(x^2 - 4)^3}

Second-order critical #’s:
• f"(x) = 0: 12x \left(x^2 + 12\right) = 0
  \quad x = 0 \quad only

New construct sign chart for f"(x):
(Critical #’s AND vertical asymptotes are cut points on sign chart.)

\[
f"(x) = \frac{12x \left(x^2 + 12\right)}{(x^2 - 4)^3}
\]
$f''(-3) = \frac{\text{O}}{\text{O}} = \text{O}, \ f''(1) = \frac{\text{O}}{\text{O}} = \text{O}$

$\ f''(-1) = \frac{\text{O}}{\text{O}} = \text{O}, \ f''(3) = \frac{\text{O}}{\text{O}} = \text{O}$

$f$ is concave down on $(-\infty, -2)$, $(0,2)$

$f$ is concave up on $(-2,0)$, $(2,\infty)$

$f$ has inflection point @ $x = 0$

(No inflection point at $x = -2$ or $x = 2$! The lines $x = \pm 2$ are vertical asymptotes.)

Graph $y = f(x)$.

Important features of Graph:

$(0,0)$: inflection point

$y = 0$: horizontal asymptote

$x = -2$: vertical asymptote

$x = 2$: vertical asymptote

Summary of Info from $f, f', f''$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\text{Dec.}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$\text{Dec.}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\text{Dec.}$</td>
</tr>
<tr>
<td>$2$</td>
<td>$\text{Dec.}$</td>
</tr>
<tr>
<td>$\infty$</td>
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