Section 4.3: Shapes of Graphs

* Be sure to review supplementary notes for Section 4.3: “Conceptual Background for Shapes of Graphs”

**Ex. 1**

Graph \( f(x) = x^3 - 12x^2 \) on \([-1, 9]\).

**Solution**:

\[
\begin{align*}
\quad f(x) &= x^3 - 12x^2 = x^2(x - 12) \\
\quad f'(x) &= 3x^2 - 24x = 3x(x - 8) \\
\quad f''(x) &= 6x - 24 = 6(x - 4)
\end{align*}
\]

1. **Information from** \( f(x) \):
   
   *(Polynomials have no asymptotes)*

2. **Information from** \( f'(x) \):
   
   First-order critical #’s:
   
   *(Recall \( f'(x) = 3x(x - 8) \))*
   
   * \( f'(x) \) does: none
   
   * \( f'(x) = 0 \): \( x = 0, x = 8 \)

Now construct sign chart for \( f'(x) \):
The shape of $f$ is shown on the number line with signs indicating the behavior of $f'$. The equation for $f'(x)$ is given as $f'(x) = 3x(x-8)$. The sign of $f'$ at specific points is tested:

- $f'(-1) = -\phantom{0} - = +$
- $f'(1) = +\phantom{0} - = -$
- $f'(9) = +\phantom{0} + = +$

$f$ is decreasing on $[0,8]$ and acceptable: $(0,8)$.

$f$ is increasing on $(-\infty,0]$, $[8,\infty)$, and acceptable: $(-\infty,0) \cup (8,\infty)$.

Local min @ $x = 8$

Local max @ $x = 0$

Information from $f''(x)$:
- Second-order critical points: (Recall $f''(x) = 6(x-4)$)
- $f''(x)$ due: none
• $f''(x) = 0 : \ x = 4$

Now construct sign chart for $f''(x)$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f''(x)$</th>
<th>Shape of $f$</th>
<th>Sign of $f''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\infty$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$\infty$</td>
<td>$+$</td>
<td></td>
</tr>
</tbody>
</table>

$f''(x) = 6(x - 4)$

$f''(0) = \infty \ - \ - = -$

$f''(5) = \infty \ + \ + = +$

$f$ is concave down on $(-\infty, 4]$

acceptable: $(-\infty, 4)$

$f$ is concave up on $[4, \infty)$

acceptable: $(4, \infty)$

Inflection point(s) @ $x = 4$

Graph $y = f(x)$:

Recall $f(x) = x^2(x-12)$ on $[-1, 9]$

Important points on graph:

<table>
<thead>
<tr>
<th>$x$-value</th>
<th>$y$-value</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-13</td>
<td>endpoint</td>
</tr>
</tbody>
</table>
Summary of Info from $f$, $f'$, and $f''$:

- **Local Max:** $(0,0)$
- **Inflection Point:** $(4,-128)$
- **Local Min:** $(8,-256)$

**Inc/dec**

- **Inc.**
- **Dec.**
- **Inc.**

**Concave Up/Down**

- **Concave Down**
- **Concave Up**
(Make sure to label important points. Graph does not have to be to scale.)

**Ex. 2**

Graph \( f(x) = x(x-2)^3 \) on \([-1, 3] \).

**Solution:**

\[
f(x) = x(x-2)^3
\]

\[
f'(x) = 1 \cdot (x-2)^3 + x \cdot 3(x-2)^2 \cdot 1
\]

\[
= (x-2)^2 (3x - 3)
\]

\[
= 2(2x-1)(x-2)^2
\]

\[
f''(x) = 2 \left[ 2 \cdot (x-2)^2 + (2x-1) \cdot 2(x-2) \cdot 1 \right]
\]

\[
= 2(x-2)(2x-2 + 2(2x-1))
\]

\[
= 12(x-2)(x-1)
\]

(Do not expand. Just use product rule.)

1. **Information from \( f(x) \):**

   (Polynomials have no asymptotes)

2. **Information from \( f'(x) \):**

   First-order critical \#'s:

   (Recall \( f'(x) = 2(2x-1)(x-2)^2 \))

   - \( f'(x) \) undefined: none
   - \( f'(x) = 0 \): \( x = \frac{1}{2}, x = 2 \)
Now construct sign chart for $f'(x)$:

- $f'(x) = 2(2x-1)(x-2)^2$
- $f'(0) = -$
- $f'(1) = +$
- $f'(3) = +$

$f$ is decreasing on $(-\infty, \frac{1}{2}]$
$f$ is increasing on $[\frac{1}{2}, \infty)$

Not two separate intervals!

Local min @ $x = \frac{1}{2}$
Local max @ nowhere

3 Information from $f''(x)$:

Second-order critical #s: (Recall $f''(x) = 12(x-2)(x-1)$)
- $f''(x)$ due: none
- $f''(x) = 0: x = 1, x = 2$
Now construct sign chart for $f''(x)$:

$$f''(x) = 12(x-2)(x-1)$$

- $f''(0) = + - - = +$
- $f''(1.5) = + - + = -$
- $f''(3) = + + + = +$

$f$ is concave down on $[1, 2]$

$f$ is concave up on $(-\infty, 1]$, $[2, \infty)$

Inflection point(s) @ $x = 1$, $x = 2$

4. Graph $y = f(x)$:

Recall $f(x) = x(x-2)^3$ on $[-1, 3]$

Important Points on Graph:

<table>
<thead>
<tr>
<th>x-value</th>
<th>y-value</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>27</td>
<td>endpoint</td>
</tr>
<tr>
<td>1/2</td>
<td>-27/16</td>
<td>local min</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>inflection point</td>
</tr>
</tbody>
</table>
Summary of Info from $f$, $f'$, and $f''$:

<table>
<thead>
<tr>
<th>Inc/dec</th>
<th>Concave up/down</th>
</tr>
</thead>
<tbody>
<tr>
<td>dec.</td>
<td>concave up</td>
</tr>
<tr>
<td></td>
<td>concave up</td>
</tr>
<tr>
<td>increasing</td>
<td>conc. down</td>
</tr>
<tr>
<td></td>
<td>conc. up</td>
</tr>
</tbody>
</table>

Inflection point $(2, 0)$

Inflection point $(1, -1)$

Local min $(\frac{1}{2}, -\frac{27}{16})$

(Make sure to label important points. Graph does not have to be to scale.)
Ex. 3

Graph \( f(x) = \frac{x}{x^2 - 4} \).

Solution:

\[ f'(x) = \frac{-(x^2+4)}{(x^2-4)^2}, \quad f''(x) = \frac{12x(x^2+12)}{(x^2-4)^3} \]

1. Information from \( f(x) \):
   - **Horizontal Asymptotes**:
     \[
     \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x}{x^2 - 4} = \lim_{x \to \infty} \left( \frac{x}{x^2} \cdot \frac{1}{1 - \frac{4}{x^2}} \right) = \lim_{x \to \infty} \left( \frac{1}{x} \cdot \frac{1}{1 - \frac{4}{x^2}} \right) = 0 \cdot \frac{1}{1-0} = 0
     \]
     \[
     \lim_{x \to -\infty} f(x) = 0 \text{ (same as previous calculation)}
     \]
     The only HA is \( y = 0 \).
   - **Vertical Asymptotes**
     (Look at where denominator is 0.
     \[ x^2 - 4 = 0 \implies x = -2 \text{ or } x = 2. \])
\[
\lim_{x \to -2^-} \frac{x}{x^2-4} = \frac{2}{0^-} \cdot \infty = -\infty \quad \text{if } x \to -2^-, \ x^2-4 > 0
\]
\[
\lim_{x \to -2^+} \frac{x}{x^2-4} = \frac{-2}{0^+} \cdot \infty = +\infty \quad \text{if } x \to -2^+, \ x^2-4 < 0
\]

D.S. of \( x = -2 \) gives \( \frac{-2}{0} \) for both. So each limit is infinite. Positive or negative?

\[
\lim_{x \to 2^-} \frac{x}{x^2-4} = \frac{2}{0^-} \cdot \infty = -\infty \quad \text{if } x \to 2^-, \ x^2-4 < 0
\]
\[
\lim_{x \to 2^+} \frac{x}{x^2-4} = \frac{2}{0^+} \cdot \infty = +\infty \quad \text{if } x \to 2^+, \ x^2-4 > 0
\]

D.S. of \( x = 2 \) gives \( \frac{2}{0} \) for both. So each limit is infinite. Positive or negative?

So \( x = -2 \) and \( x = 2 \) are VA's.

2) Information from \( f'(x) \):

Recall: \( f'(x) = \frac{-(x^2+4)}{(x^2-4)^2} \)
First-order critical #’s

- \( f'(x) \) due: none \( (x = \pm 2 \) not in domain of \( f \), so not critical #) 

- \( f'(x) = 0: \ - (x^2 + 4) = 0 \)
  - no solution

New construct sign chart for \( f'(x) \):
- Critical #’s AND vertical asymptotes are cut points on sign chart.

\[ f'(x) = \frac{-(x^2 + 4)}{(x^2 - 4)^2} \]

- \( f'(-3) = \frac{-\bigcirc}{\bigcirc} = 0 \)
- \( f'(0) = \frac{-\bigcirc}{\bigcirc} = 0 \)
- \( f'(3) = \frac{-\bigcirc}{\bigcirc} = 0 \)

\( f \) is decreasing on \( (-\infty, -2), (-2, 2), (2, \infty) \)
$f$ is increasing on no interval
$f$ has local min @ no x-value
$f$ has local max @ no x-value

3) Information from $f''(x)$:

Recall: 
$$f''(x) = \frac{12x(x^2+12)}{(x^2-4)^3}$$

Second-order critical #’s:
- $f''(x)$ due: none
- $f''(x) = 0$: $12x(x^2+12) = 0$
  $$x = 0$$ only

New construct sign chart for $f''(x)$:
(Critical #’s AND vertical asymptotes are cut points on sign chart.)

Shape of $f$
Sign of $f''$
Test point

$f''(x) = \frac{12x(x^2+12)}{(x^2-4)^3}$
\[ f''(-3) = \frac{-}{\color{red}+} = \color{red}{-}, \quad f''(1) = \frac{\color{red}+}{-} = \color{red}{-} \]
\[ f''(-1) = \frac{-}{\color{red}+} = \color{red}{+}, \quad f''(3) = \frac{\color{red}+}{\color{red}+} = \color{red}{+} \]

\( f \) is concave down on \((-\infty, -2), (0, 2)\)
\( f \) is concave up on \((-2, 0), (2, \infty)\)
\( f \) has inflection point @ \( x = 0 \)

(Number inflection point at \( x = -2 \) or \( x = 2 \)!
The lines \( x = \pm 2 \) are vertical asymptotes)

4. Graph \( y = f(x) \).

**Important features of Graph:**

\((0, 0)\): inflection point
\( y = 0 \): horizontal asymptote
\( x = -2 \): vertical asymptote
\( x = 2 \): vertical asymptote

Summary of Info from \( f, f', f'' \):

\[
\begin{array}{cccccc}
& -\infty & & -2 & & 0 & & 2 & & \infty \\
\text{Inc/Dec.} & \text{dec.} & \leftarrow & \text{dec.} & \rightarrow & \text{dec.} \\
\end{array}
\]
**Ex. 4**

Graph \( f(x) = x^2 e^{-x} \).

**Solution:**

\[
\begin{align*}
  f'(x) &= x^2 e^{-x}(-1) + 2xe^{-x} = (2x-x^2)e^{-x} \\
  f''(x) &= (2x-x^2)e^{-x}(-1) + (2-2x)e^{-x} = (x^2-4x+2)e^{-x} \\
  &= ((x-2)^2-2)e^{-x}
\end{align*}
\]

1. **Information from \( f(x) \):**
   - Horizontal asymptotes
     \[
     \lim_{{x \to -\infty}} (x^2 e^{-x}) = (\infty)(0) = 0
     \]
\[
\lim_{x \to \infty} \left( x^2 e^{-x} \right) = \lim_{x \to \infty} \left( \frac{x^2}{e^x} \right) = \lim_{x \to \infty} \left( \frac{2x}{e^x} \right)
\]

indeterminate

\[
\lim_{x \to \infty} \left( \frac{2x}{e^x} \right) = \frac{2}{\infty} = 0
\]

The only H.A. is \( y = 0 \).

- Vertical asymptotes
  No V.A. since \( f(x) \) is continuous for all \( x \).

2) Information from \( f'(x) \):

Recall: \( f'(x) = x (2-x) e^{-x} \)

- Cut points for sign chart for \( f'(x) \):
  - \( f'(x) \) due: none
  - \( f'(x) = 0 \): \( x = 0 \), \( x = 2 \)

- Sign chart for \( f'(x) \):

  ![Sign chart for f'(x)](image)

  Shape of \( f \)

  Sign of \( f' \)

  Test point
\[ f'(x) = x(2-x)e^{-x} \]

\[ f'(-1) = -1 \quad + \quad + = + \]

\[ f'(1) = + \quad + \quad + = + \]

\[ f'(3) = + \quad - \quad + = - \]

\( f \) is decreasing on: \((-\infty, 0], [2, \infty)\)

\( f \) is increasing on: \([0, 2]\)

\( f \) has a local min at \( x = 0 \)

\( f \) has a local max at \( x = 2 \)

3) Information from \( f''(x) \):

Recall: \( f''(x) = ((x-2)^2 - 2)e^{-x} \)

- Cut points for sign chart for \( f''(x) \)
  - \( f''(x) \) due: none
  - \( f''(x) = 0: \ (x-2)^2 - 2 = 0 \)
    \[ x = 2 + \sqrt{2} \quad \text{or} \quad x = 2 - \sqrt{2} \]

- Sign chart for \( f''(x) \):

\[
\begin{array}{cccccc}
0 & - & 2 - \sqrt{2} & + & 2 + \sqrt{2} & - \\
\end{array}
\]

Shape of \( f \)

Sign of \( f'' \)

Test point
\[ f''(x) = ((x-2)^2 - 2) e^{-x} \]
\[ f''(0) = (2^2 - 2) \bigcirc = \bigcirc \]
\[ f''(2) = (0 - 2) \bigcirc = \bigcirc \]
\[ f''(4) = (2^2 - 2) \bigcirc = \bigcirc \]

\( f \) is concave down on: \([2 - \sqrt{2}, 2 + \sqrt{2}]\)

\( f \) is concave up on: \((-\infty, 2 - \sqrt{2}],[2 + \sqrt{2}, \infty)\)

\( f \) has inflection points at \( x = 2 - \sqrt{2} \) and \( x = 2 + \sqrt{2} \)

4. **Graph** \( y = f(x) \)

- Important features of graph:
  - \( x = 0 \): local min
  - \( x = 2 - \sqrt{2} \): inflection point
  - \( x = 2 \): local max
  - \( x = 2 + \sqrt{2} \): inflection point
  - \( y = 0 \): horizontal asymptote (as \( x \to \infty \))

- Summary of Info from \( f, f', f'' \):

<table>
<thead>
<tr>
<th>(-\infty)</th>
<th>0</th>
<th>2 - ( \sqrt{2} )</th>
<th>2</th>
<th>2 + ( \sqrt{2} )</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>inc./dec.</td>
<td>dec.</td>
<td>inc.</td>
<td>dec.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>