Section 4.2: Mean Value Theorem

What types of functions are continuous but not differentiable? (c.f., 3.1)

(In 3.1, we covered how to recognize such functions graphically. The following notes describe how to recognize such functions analytically. These notes were summarized at the end of notes for 3.1.)

There are three categories to look out for:

1. **Absolute Value**: \( f(x) = |x| \)

   The function \( f \) is not differentiable at \( x=0 \)

   The graph of \( f \) has a sharp corner at \( x=0 \).

   So the function \( h(x) = |g(x)| \) is (possibly) not differentiable when \( g(x) = 0 \).
Ex: The function $h(x) = |x^2 - 4|$ is not differentiable at $x = -2$ and $x = 2$ ($x^2 - 4 = 0 \implies x = \pm 2$)

2) Power Functions: $f(x) = x^n$ (Exponents less than 1) ($0 < n < 1$)

The function $f$ is not differentiable at $x = 0$. The graph of $f$ has a cusp or vertical tangent line at $x = 0$.

So the function $h(x) = g(x)^n$ is (possibly) not differentiable when $g(x) = 0$.

Ex: The function $h(x) = (x^2 - 4)^{2/3}$ is not differentiable at $x = -2$ and $x = 2$ ($x^2 - 4 = 0 \implies x = \pm 2$)
Similarly, \( h(x) = (x^2 - 4)^{1/3} \) is not differentiable at \( x = -2 \) and \( x = 2 \); \( h(x) = (x^2 - 4)^{1/2} \) is also not differentiable at \( x = -2 \) and \( x = 2 \).

\[ y = (x^2 - 4)^{2/3} \quad y = (x^2 - 4)^{1/3} \quad y = (x^2 - 4)^{1/2} \]

3 \textbf{Piecewise-Defined Functions}  
(assume each piece is continuous)

Very often (but not always), a piecewise-defined function is not differentiable at the transition points.

\[ y = \begin{cases} 
-2x & \text{if } x < 0 \\
 x^2 & \text{if } x \geq 0 
\end{cases} \]  
Not diff. at \( x = 0 \)

\[ y = \begin{cases} 
1 - \cos(x) & \text{if } x < 0 \\
 x^2 & \text{if } x \geq 0 
\end{cases} \]  
Differentiable at \( x = 0 \)
Thm: (Mean Value Theorem, MVT)
Suppose \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Then there exists \( c \) in \((a, b)\) with

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

**Graphical Interpretation of MVT**

\[\text{slope} = f'(c)\]

\[\text{slope} = \frac{f(b) - f(a)}{b - a}\]

**Ex. 1**

Let \( f(x) = x^{2/3} \). For each interval, determine whether the hypotheses of the MVT are satisfied. If yes, find
all values of $c$ guaranteed to exist by the MVT.
(a) $[-1, 1]$
(b) $[0, 8]$

Solution:
• Where is $f$ continuous?
Power functions are cont. on their domain, so $f$ is cont. on $(-\infty, \infty)$.
So the continuity hypothesis in MVT is always satisfied for $f$.

• Where is $f$ differentiable?
Everywhere except $x = 0$. $(0 < \frac{2}{3} < 1)$
So any open interval containing $x = 0$ does not satisfy the MVT hypothesis.
(a) Since $0$ is in $(-1, 1)$, hypotheses of MVT are not satisfied.
(b) Since $0$ is not in $(0, 8)$, hypotheses of MVT are satisfied. So there exists
c in \((0, 8)\) such that
\[f'(c) = \frac{f(8) - f(0)}{8 - 0}\]

Now calculate \(f'(c)\) and solve this equation for \(c\).

\[
\frac{2}{3} c^{-\frac{1}{3}} = \frac{8^{2/3} - 0^{2/3}}{8 - 0}
\]

\[
\frac{2}{3} c^{-\frac{1}{3}} = 8^{-\frac{1}{3}}
\]

\[
c = \left(\frac{3}{2}\right)^{-3}. 8 = \frac{64}{27}
\]

**Important Special Case of MVT**

**Thm. (Rolle's Theorem)**

Suppose \(f\) is continuous on \([a, b]\),
differentiable on \((a, b)\), and \(f(a) = f(b)\). Then there exists \(c\) in \((a, b)\) with

\[f'(c) = 0 \quad \left(= \frac{f(b) - f(a)}{b - a}\right)\]