A differentiable function is "locally linear". In other words, the tangent line to $f(x)$ at $x = a$ is an approximation of $f(x)$ near $x = a$.

**Analytical Intuition:**

By the definition of derivative,

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

if $\Delta x$ is near 0 (i.e. $\Delta x$ is small)
Then from definition of limit,
\[ f'(a) \approx \frac{\Delta f}{\Delta x}. \]

So if \( \Delta x \) is small,
\[
f'(a) \approx \frac{\Delta f}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x}
\]

→ Algebra to rearrange.....

\[
f(a + \Delta x) \approx f(a) + f'(a) \Delta x
\]

→ tangent line approximation !
\[
f(x) \approx f(a) + f'(a)(x-a)
\]

**Tangent Line**

**Terminology and Notation**

1. **Linearization**
   * synonymous with tangent line
   * linearization is a function of \( x \)

**Ex:** Find tangent line to \( f(x) = x^2 \) at \( x = 1 \).
at $x = 3$.

**Solution:** $y - 9 = 6(x - 3)$

Find linearization to $f(x) = x^2$ at $x = 3$.

**Solution:** $L(x) = 9 + 6(x - 3)$

2. **Differentials** *(More in chapter 5)*

   * Right now this is just notation
   * If $y = f(x)$, then
     $$dy = f'(x) \, dx$$

**Example:**

Calculate the differential of $f(x) = x^2$ at $x = 3$.

**Solution:**

\[
\begin{align*}
dy &= 2x \, dx \\
\text{sub } x = 3
\end{align*}
\]

Small change in $y = x^2$ near $x = 3$ = $6$ times small change in $x$. 
Ex. 1

Use a linear approximation to estimate the value of

\[ \tan \left( \frac{\pi}{4} + 0.01 \right) \]

Solution:
The phrase “linear approximation” indicates a tangent line approximation.

Put \( f(x) = \tan(x) \). Then find tangent line to \( f(x) \) at \( x = \frac{\pi}{4} \).

Why choose \( x = \frac{\pi}{4} \)?

- \( x = \frac{\pi}{4} \) is close to \( x = \frac{\pi}{4} + 0.01 \)
- The values of \( f(x) = \tan(x) \) and its derivatives are known exactly at \( x = \frac{\pi}{4} \).
- Need to get approximation without a calculator.

\[ f\left( \frac{\pi}{4} \right) = \tan \left( \frac{\pi}{4} \right) = 1 \]
\[ f(x) = \tan(x) \]
\[ f'(x) = \sec(x)^2 \]
\[ f'(\frac{\pi}{4}) = \sec(\frac{\pi}{4})^2 = (\sqrt{2})^2 = 2 \]

So the linearization of \( f(x) \) at \( x = \frac{\pi}{4} \) is

\[ l(x) = 1 + 2\left(x - \frac{\pi}{4}\right) \]

**Note:** If \( x \) is near \( \frac{\pi}{4} \), then

\[ \tan(x) \approx 1 + 2\left(x - \frac{\pi}{4}\right) \]

So our approximation gives

\[ \tan\left(\frac{\pi}{4} + 0.01\right) \approx 1 + 2\left(\frac{\pi}{4} + 0.01 - \frac{\pi}{4}\right) \]
\[ = 1 + 2(0.01) \]
\[ = 1.02 \]

**Ex. 2**

Use a linear approximation to estimate the value of \( \tan(18.4^\circ) \).
estimate the value of $18^{1/4}$.

Solution:

Put $f(x) = x^{1/4}$. Then find the tangent line at $x = 16$.

$f(16) = 16^{1/4} = 2$

$f'(x) = \frac{1}{4} \cdot x^{-3/4} = \frac{1}{4x^{3/4}}$

$f'(16) = \frac{1}{4 \cdot (16^{1/4})^3} = \frac{1}{4 \cdot 2^3} = \frac{1}{32}$

So the linearization of $f(x)$ at $x = 16$ is

$L(x) = 2 + \frac{1}{32}(x-16)$

Note: If $x$ is close to 16, then

$x^{1/4} \approx 2 + \frac{1}{32}(x-16)$

So our approximation gives

$18^{1/4} \approx 2 + \frac{1}{32}(18-16) = 2 + \frac{1}{16} = 2.0625$
Use a linear approximation to estimate the value of $\sqrt{3.9}$.

Solution:

Put $f(x) = \sqrt{x}$. Then find tangent line at $x = 4$.

$$f(4) = \sqrt{4} = 2$$
$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$f'(4) = \frac{1}{2 \cdot 2} = \frac{1}{4}$$

Linearization is

$$L(x) = 2 + \frac{1}{4}(x-4)$$

Note: If $x$ is near 4, then

$$\sqrt{x} \approx 2 + \frac{1}{4}(x-4)$$

So our approximation gives
\[
\sqrt{3.9} \approx 2 + \frac{1}{4}(3.9-4) = 2 + \frac{1}{4}(-0.1) = 2 - \frac{1}{40} = 1.975
\]

\[
\frac{1}{20} = 0.05, \text{ so } \frac{1}{40} = 0.025
\]

---

**Terminology from Economics**

* \( x = \# \text{ of units produced/sold} \)
  (we assume entire inventory is sold unless stated otherwise)

* \( R(x) = \text{total revenue from selling } x \text{ units} \)

\[
R(x) = (\# \text{ of units sold}) \cdot (\text{price per unit})
\]

\[
R(x) = x \cdot p(x)
\]

* \( C(x) = \text{total cost of producing } x \text{ units} \)

**Marginal quantities:**
*Marginal revenue is the additional revenue gained from selling one more unit if \( x \) units have already been sold.

\[
MR(x) = R(x+1) - R(x)
\]

- Revenue from selling \( x+1 \) units
- Revenue from selling \( x \) units

*Marginal cost is the additional cost incurred from producing one more unit if \( x \) units have already been produced.

\[
MC(x) = C(x+1) - C(x)
\]

- Cost from producing \( x+1 \) units
- Cost from producing \( x \) units

Standard approximation of marginal “\( Q \)”

Suppose \( Q(x) \) represents some
Suppose \( q \) is a function of quantity (revenue, cost, profit, etc.).

Then “marginal \( q \)” is

\[
MQ(x) = Q(x+1) - Q(x)
\]

The tangent line to \( Q(x) \) at \( x = a \) is

\[
L(x) = Q(a) + Q'(a)(x-a)
\]

Note: If \( x \) is near \( a \), then

\[
Q(x) \approx Q(a) + Q'(a)(x-a)
\]

Now put \( x = a + 1 \), we get

\[
Q(a+1) \approx Q(a) + Q'(a)(a+1 - a) = 1
\]

So we have

\[
MQ(a) = Q(a+1) - Q(a) \approx Q'(a)
\]

So we can use this for marginal revenue and marginal cost.
If a factory produces \( x \) units, the total cost is
\[
C(x) = \frac{1}{8} x^2 + 3x + 98
\]
and the selling price per unit is
\[
p(x) = \frac{1}{3} (75 - x)
\]
(We assume \( 0 \leq x \leq 50 \).

(a) What is the (exact) marginal cost as a function of \( x \)? Same for marginal revenue.
(b) Use marginal analysis (i.e., use a linear approximation) to estimate the cost of producing the 9th unit. Same for additional revenue from 9th unit.

Solution:

(a) By definition, we have

\[ MC(x) = C(x+1) - C(x) \]

Recall: \( C(x) = \frac{1}{8}x^2 + 3x + 98 \)

\[
= \left[ \frac{1}{8}(x+1)^2 + 3(x+1) + 98 \right] - \left[ \frac{1}{8}x^2 + 3x + 98 \right]
\]

\[ = C(x+1) - C(x) \]

\[ = \frac{x}{4} + 3 + \frac{1}{8} \]

For marginal revenue, we have...

\[ R(x) = x \cdot p(x) = \frac{x}{2} (75-x) = 25x - \frac{1}{2}x^2 \]
Recall: \( p(x) = \frac{1}{3}(75-x) \)

\[
MR(x) = R(x+1) - R(x)
= \left[25(x+1) - \frac{1}{3}(x+1)^2\right] - \left[25x - \frac{1}{3}x^2\right]
= R(x+1) - R(x)
= 25 - \frac{2x}{3} - \frac{1}{3}
\]

(b) “Find approximate cost of 9th unit. Same for revenue from 9th unit.”

Before that, note the exact quantities:

- exact cost of 9th unit:
  \[
  MC(8) = \left(\frac{x}{4} + 3 + \frac{1}{8}\right)\bigg|_{x=8} = 5 + \frac{1}{8}
  \]

- exact revenue from 9th unit:
  \[
  MR(8) = \left(25 - \frac{2x}{3} - \frac{1}{3}\right)\bigg|_{x=8} = \frac{59}{3} - \frac{1}{3}
  \]

For the approximations, we use the standard approximations.
\[ MC(x) \approx C'(x) = \frac{1}{4}x + 3 \]

Recall: \[ C(x) = \frac{1}{8}x^2 + 3x + 98 \]

\[ MC(8) \approx C'(8) = 5 \]

\[ MR(x) \approx R'(x) = 25 - \frac{2x}{3} \]

Recall: \[ R(x) = 25x - \frac{1}{3}x^2 \]

\[ MR(8) \approx R'(8) = \frac{59}{3} \]

---

**Error Propagation (Physics)**

Suppose we measure some quantity to have a value of \( x_0 \) and the maximum uncertainty in this measurement is \( \delta x \).

What is the approximate maximum uncertainty in the derived Quantity \( Q(x_0) \)?
exact value of $x$: $x_0 + \Delta x$
measured value of $x$: $x_0$

exact value of $Q$: $Q(x_0 + \Delta x)$
measured value of $Q$: $Q(x_0)$

So the difference between the exact and measured value of $Q$ is:

$$Q(x_0 + \Delta x) - Q(x_0) \approx Q'(x_0) \Delta x$$

These values of $x$ are near each other.

Recall: $Q(x) \approx Q(a) + Q'(a) (x-a)$

Put $a = x_0$

$$x = x_0 + \Delta x$$

Summary: If we measure $x_0$ with max uncertainty $\Delta x$, then the approximate max uncertainty $\Delta Q$ is
Suppose the radius of a disk is measured to be 2m with a maximum uncertainty of 1cm. This measurement is then used to calculate the area of the disk.

(a) What is the maximum uncertainty in the area? (Use a linear approximation to estimate.)

(b) What is the estimated maximum relative error in the area?

Solution:

(a) In the language of the preamble, we have

\[ x_0 = 2 \]
\[ \delta x = 0.01 \] (1cm = 0.01m)

We want to estimate error in the area.
the area:

\[ A(x) = \pi x^2 \]

\[ A'(x) = 2\pi x \]

So from our formula, we have

(Recall: \( \delta Q \approx |Q'(x_0)| \delta x \))

\[ \delta A \approx |A'(2)| \delta x = 2\pi \cdot 2 \cdot 0.01 \]

\[ \delta A \approx 0.04\pi \text{ m}^2 \]

(This means the area is off by at most \( 0.04\pi \text{ m}^2 \) if we use the imperfect measurement of the radius.)

(b) By definition, the relative error in some quantity \( Q \) is:

\[ \frac{\delta Q}{Q} \]

So our estimate of the relative error in the area is

\[ \frac{\delta A}{A(x_0)} = \frac{0.04\pi}{\pi(2)^2} = \frac{0.04\pi}{4\pi} = 0.01 = 1\% \]