Section 3.7: Related Rates

Note: All variables are assumed to be functions of time $t$.

**Ex. 1**

A ladder of length $L = 10\,\text{ft}$ is leaning against a wall. Suppose the bottom of the ladder slides away from the wall at $2\,\text{ft/s}$. How fast and in what direction is the top of the ladder moving when the top is 8 ft from the ground?

**Solution:**
Given \( \frac{dx}{dt} = 2 \)
Want \( \frac{dy}{dt} = ?? \) When \( y = 8 \)

Very important: Always find equations and relations that hold for all time first. Then use information valid for a specific time or for this specific situation.

\[
x^2 + y^2 = 100 \quad \leftarrow \text{valid for all time}
\]

\[
(x(t)^2 + y(t)^2 = 100)
\]

Use implicit differentiation. (Diff. wrt time \( t \).) This will introduce \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \) into the problem.

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0
\]
\[ \frac{dx}{dt} + \frac{dy}{dt} = 0 \]

Now we collect our equations, which hold for all time.

\begin{align*}
    x^2 + y^2 &= 100 \\
    x \frac{dx}{dt} + y \frac{dy}{dt} &= 0
\end{align*}

Now we can use the information specific to this problem!

Substitute \( \frac{dx}{dt} = 2 \) and \( y = 8 \).

\begin{align*}
    x^2 + 64 &= 100 \\
    2x + 8 \frac{dy}{dt} &= 0
\end{align*}

These equations are valid for one specific time: when \( y = 8 \).

Now solve for \( \frac{dy}{dt} \). We get \( x = 6 \) from first equation. Then we get

\( \frac{dy}{dt} = -\frac{1}{4} x \).
From first equation: Then second equation gives

\[ 12 + 8 \frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{3}{2} \text{ ft/s} \]

**Final answer.**

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**Reminder:** We use implicit differentiation or really just chain rule.

\[
\frac{d}{dt}(x^2) = \frac{d}{dt}(x(t))^2 = 2(x(t)) \cdot \frac{dx}{dt}
\]

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**General Procedure for Related Rates:**

1. Draw diagram, label variables
2. Write down all equations relating variables, which hold for all time.
3. Differentiate all equations with respect to time.
4) Substitute given values, which hold for the specific time only.
5) Solve for desired values

Ex. 2
Lighthouse at P rotates at 1 rev/min. Point Q is directly across on straight beach. How fast is light beam on beach sweeping across when it passes through Q?

Solution:
\[ x = \text{distance from } Q \text{ to position of light beam on beach} \]
\[ \theta = \angle QPR, \text{ angle light beam makes with vertical.} \]

\[ \frac{d\theta}{dt} = \frac{1 \text{ rev}}{\text{min}} \quad \text{Want} \quad \frac{dx}{dt} = ?? \quad \text{when } x = 0 \]

Careful! Our derivative rules are true only if angles are measured in radians. So we should convert the units to radians/min.

(So we will use \( \frac{d\theta}{dt} = 2\pi \frac{\text{rad}}{\text{min}} \).)
Now we find an equation that relates $x$ and $\theta$.

$1000 \tan(\theta) = x$ \hspace{1cm} (1)

Now differentiate wrt. $t$.

$1000 \sec(\theta)^2 \cdot \frac{d\theta}{dt} = \frac{dx}{dt}$ \hspace{1cm} (2)

Equations (1) and (2) hold for all $t$.

Now we finally substitute info that holds for the specific time.

So $\frac{d\theta}{dt} = 2\pi$ and $x = 0$.

$1000 \tan(\theta) = 0$ \hspace{1cm} (3)

$2000\pi \sec(\theta)^2 = \frac{dx}{dt}$ \hspace{1cm} (4)

hold at specific time only.
From Equation (3), we get $\Theta = 0$. Now sub $\Theta = 0$ into (4).

\[
\frac{dx}{dt} = 2000\pi \sec(0)^2 = 2000\pi \frac{\text{ft}}{\text{min}}.
\]

\[
\sec(0) = \frac{1}{\cos(0)} = 1
\]

: thinking:

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**Ex. 3**

Inverted conical tank has height 2m and radius 2m at top. Water flows into tank at a rate of $2\text{ m}^3/\text{min}$. How fast is the water level rising when the water level is 2m?

**Cross-section of tank**
Solution:

$h$ = water level at time $t$.

$R = (\text{constant})$ radius of tank

$H = (\text{constant})$ height of tank

$V = \text{volume of water at time } t$

**Given** \[
\frac{dV}{dt} = 2
\]

**Want** \[
\frac{dh}{dt} = ?? \text{ when } h = 2
\]
Now find an equations that relates the variables \((h \text{ and } V)\).

But the value of \(h\) is not enough to find the value of \(V\).
We need to introduce a third variable \(r\) ...

\[
 r = \text{radius of water at time } t.
\]

We do have an equation for \(h, r, \text{ and } V\).

\[
 V = \frac{\Pi}{3} r^2 h.
\]

We have too many variables!

Is there another equation that relates \(h, r, \text{ and } V\) (or just some subset)?

We can use similar triangles to find an equation for \(h\).
Using similar triangles:

\[
\frac{\text{little radius}}{\text{big radius}} = \frac{\text{little height}}{\text{big height}}
\]

\[
\frac{r}{2} = \frac{h}{3}
\]

\[\Rightarrow r = \frac{2}{3} h\]

So we currently have the following:

\[V = \frac{\pi}{3} r^2 h\]

Before differentiating, we can eliminate \(r\) in favor of \(h\).

\[V = \frac{4\pi}{3} h^3\]
Now we differentiate with respect to \( t \):

\[
\frac{dV}{dt} = \frac{4\pi}{9} h^2 \cdot \frac{dh}{dt}
\]

Now we substitute info for our specific time \((\frac{dV}{dt} = 2, h = 2)\):

\[
V = \frac{4\pi}{27} \cdot 8
\]

\[
2 = \frac{16\pi}{9} \cdot \frac{dh}{dt}
\]

Now solve for \( \frac{dh}{dt} \):

\[
\frac{dh}{dt} = \frac{18}{16\pi} \text{ m/min}
\]