Section 3.5: The Chain Rule

How do we differentiate ... 

\[ f(x) = \sin(x^2) \]
\[ g(x) = \ln(\cos(x) - 3x^2) \]
\[ h(x) = e^{x^2-2} \cdot \tan(x+4) \]

The Chain Rule is for differentiating compositions of functions.

**Thm:** (Chain Rule)

If \( f \) and \( g \) are differentiable,

\[ (f \circ g)'(x) = (f' \circ g)(x) \cdot g'(x) \]

\[ \frac{d}{dx} \left( f(g(x)) \right) = f'(g(x)) \cdot g'(x) \]

- derivative of outside function evaluated at inside function
derivative of inside function
**Ex. 1**

Find \( h'(x) \)

\[
h(x) = \sin(x^2)
\]

**Solution:**

outside: \( f(x) = \sin(x) \)
inside: \( g(x) = x^2 \)

\[
h'(x) = \cos(x^2) \cdot 2x
\]

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**Ex. 2**

Find \( h'(x) \).

\[
h(x) = e^{3x+4}
\]

**Solution:**

outside: \( f(x) = e^x \)
inside: \( g(x) = 3x + 4 \)

\[
h'(x) = e^{3x+4} \cdot 3 = 3e^{3x+4}
\]

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**Ex. 3**

Find \( h'(x) \).
Find \( h(x) \)

\[ h(x) = \ln(x^3 + x) \]

**Solution:**

outside: \( f(x) = \ln(x) \)

inside: \( g(x) = x^3 + x \)

\[ h'(x) = \frac{1}{x^3 + x} \cdot (3x^2 + 1) \]

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**Ex. 4**

Find \( h'(x) \).

\[ h(x) = \sin(e^x) \cos(x) \]

**Solution:**

Start with product rule.

\[ h'(x) = \left( \sin(e^x) \right)' \cos(x) + \sin(e^x)(-\sin(x)) \]

chain rule on this term only

**Scratch Work:**
\[
\frac{d}{dx} \left( \sin(e^x) \right) = \cos(e^x) \cdot e^x
\]

**outside:** \( f(x) = \sin(x) 
**inside:** \( g(x) = e^x \)

**Final answer:**
\[
h'(x) = \cos(e^x) \cdot e^x \cdot \cos(x) - \sin(e^x) \sin(x)
\]

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**Ex. 5**

Find \( h'(x) \)

\[
h(x) = \sqrt{\frac{x^3}{1-x}} = \left( \frac{x^3}{1-x} \right)^{\frac{1}{2}}
\]

**Solution:**

Use chain rule and power rule first.

\[
h'(x) = \frac{1}{2} \left( \frac{x^3}{1-x} \right)^{-\frac{1}{2}} \cdot \frac{d}{dx} \left( \frac{x^3}{1-x} \right)
\]

this term arises from chain rule
Scratch Work:

\[
\frac{d}{dx} \left( \frac{x^3}{1-x} \right) = \frac{(1-x) \cdot 3x^2 - x^3 \cdot (-1)}{(1-x)^2}
\]

Quotient Rule

Final answer:

\[
h'(x) = \frac{1}{2} \left( \frac{x^3}{1-x} \right)^{-1/2} \cdot \frac{(1-x) \cdot 3x^2 - x^3 \cdot (-1)}{(1-x)^2}
\]

Ex. 6

Find \( h'(x) \).

\[
h(x) = \ln (\tan (e^x))
\]

Solution:

\[
h'(x) = \frac{1}{\tan(e^x)} \cdot \frac{d}{dx} (\tan(e^x))
\]

first application of chain rule
\[ h'(x) = \frac{1}{\tan(e^x)} \cdot \sec(e^x)^2 \cdot \frac{d}{dx}(e^x) \]

second application of chain rule

\[ h'(x) = \frac{1}{\tan(e^x)} \sec(e^x)^2 \cdot e^x \]

your textbook/ Math XL writes this as

"Sec^2(e^x)"

Ex. 7

Let \( f(x) = x \sqrt{1-3x} \). Find the \( x \)-coordinates of points where the tangent line is horizontal.

**Solution:**

Since \( f'(x) \) gives us slopes of tangent lines and horizontal lines have a slope of zero, we need to find the values of \( x \) where \( f'(x) = 0 \).
have slope 0, we must solve the equation “\( f'(x) = 0 \)”.\\

\[
f(x) = x \cdot (1-3x)^{\frac{1}{2}}
\]

\[
0 = f'(x) = 1 \cdot (1-3x)^{\frac{1}{2}} + x \cdot \frac{d}{dx} (1-3x)^{\frac{1}{2}}
\]

use chain rule

dervative of inside function

\[
0 = (1-3x)^{\frac{1}{2}} + x \cdot \frac{1}{2} (1-3x)^{-\frac{1}{2}} \cdot (-3)
\]

Solve for \( x \).

Multiply by \( 2(1-3x)^{\frac{1}{2}} \)

\[
0 = 2(1-3x) - 3x
\]

\[
0 = 2 - 6x - 3x
\]

\[
x = \frac{2}{9}
\]
Check: Is $x = \frac{2}{9}$ in domain of $f(x) = x\sqrt{1-3x}$? Yes. So final answer is $x = \frac{2}{9}$.

Ex. 8

A coffee vendor has collected data on the price of coffee in her store over the last year. The price of coffee $t$ weeks since when the data collection began was

$$p(t) = 0.02t^2 - 0.1t + 6$$
dollars per pound.

For each part, you must give correct units as part of your answer.

(a) How much did the price of one pound of coffee increase in the first ten weeks after the data collection began?

(b) What was the average rate at which the price of one pound of coffee changed over the same ten-week period mentioned in part (a)?

\[ p(t) = 0.02t^2 - 0.1t + 6 \]

(a) \[ \Delta p = p(10) - p(0) \]
\[ = 0.02(100) - 0.1(10) + 6 - 6 \]
\[ = 2 - 1 + 6 - 6 \]
\[ = 1 \text{ dollar} \]

(b) \[ \frac{\Delta p}{\Delta t} = \frac{p(10) - p(0)}{10 - 0} = \frac{1}{10} \text{ dollars/week} \]
The vendor also found that, in a given week, the local consumers bought approximately

\[ D(p) = \frac{2500}{p^2 + 1} \]

pounds of coffee when the price was \( p \) dollars per pound. That is, \( D \) is the *weekly demand* of the consumers.

(c) Calculate \( D'(7) \) and explain its precise meaning in the given context.

(d) At what rate was the weekly demand for coffee changing with respect to time exactly ten weeks after data collection began?

\[ D(p) = \frac{2500}{p^2 + 1} = 2500 \left( p^2 + 1 \right)^{-1} \]

\[ (c) \ D'(p) = 2500 \left( -1 \right) \left( p^2 + 1 \right)^{-2} \cdot 2p \]

\[ D'(p) = \frac{-5000p}{(p^2+1)^2} \]

\[ D'(7) = \frac{-35000}{50^2} = \frac{-35000}{2500} = \frac{-3500}{25} = -14 \]

What are the units of \( D'(7) \)?

\[ D'(p) = \lim_{\Delta p \to 0} \frac{\Delta D}{\Delta p} = \lim_{\Delta p \to 0} \frac{D(p+\Delta p) - D(p)}{\Delta p} \]
The units of \( D'(p) \) are

\[
\frac{\text{units of } D}{\text{units of } p} = \frac{\text{pounds}}{\text{dollars}}
\]

So \( D'(7) = -14 \) pounds per dollar.

So what is the meaning?

Hint: Think about tangent lines!

If the price is already \( \$7 \) for one pound of coffee, then if we increase price by \( \Delta p \) dollars, the weekly demand decreases by
(d) Question is asking you to calculate

\[
\frac{dD}{dt} \bigg|_{t=10}
\]

Demand is really a function of time \( t \) via the price \( p \).

\[
D = D(p(t))
\]

So \( D \) is a composition of functions! **Chain Rule**!

Chain Rule gives the following:

\[
\frac{dD}{dt} = \frac{d}{dt} \left( D(p(t)) \right)
\]

\[
= D'(p(t)) \cdot p'(t)
\]

Now substitute \( t = 10 \).
\[
\frac{dD}{dt} \bigg|_{t=10} = D'(p(10)) \cdot p'(10)
\]
\[
= D'(7) \cdot p'(10)
\]
\[
= -14 \cdot p'(10)
\]
\[
p(t) = 0.02t^2 - 0.1t + 6
\]
\[
p'(t) = 0.04t - 0.1
\]
\[
p'(10) = 0.4 - 0.1 = 0.3
\]
\[
= -14 \cdot (0.3)
\]
\[
= -4.2
\]

The units are pounds per week.