Section 3.4. Rectilinear Motion

Some definitions

Given a function \( f(x) \) on the interval \( [a, b] \):

- Average rate of change over \( [a,b] \):
  \[
  \frac{f(b) - f(a)}{b - a} = \frac{\Delta y}{\Delta x}
  \]

- Instantaneous rate of change at \( x = a \):
  \[ f'(a) \]

Q: What are the units of these numbers?
A: Rates of change have units of

\[
\frac{\text{units of } f}{\text{units of } x}
\]

Ex. 1

Let \( f(x) = -\frac{1}{4}x^4 + \frac{1}{3}x^3 + 500x \)
be the number of iPads a worker
produces up until \( x \) hours after arriving at work at 6AM. (Valid for \( 0 \leq x \leq 8 \).)

(a) What is the average rate of production between 7AM and 8AM?

(b) What is the rate of production at 7AM?

(c) How many iPads are produced between 7AM and 8AM?

Solution:

(a) 7AM: \( x = 1 \), 8AM: \( x = 2 \)

\[
\frac{\Delta f}{\Delta x} = \frac{f(2) - f(1)}{2 - 1} = 495.3 \frac{\text{iPads}}{\text{hr}}
\]

(b) We need to compute \( f'(1) \).

\[
f'(x) = -x^3 + x^2 + 500
\]

\[
f'(1) = -1 + 1 + 500 = 500 \frac{\text{iPads}}{\text{hr}}
\]

(c) \( f(1) \): iPads produced between 6AM and 7AM.

\( f(2) \): iPads produced between 7AM and 8AM.
f(2) - f(1): iPads produced between 7am and 8am

\[ f(2) - f(1) = 495.3 \text{ iPads} \]

**Rectilinear Motion**

\[ x(t) : \text{position of particle along x-axis at time } t. \]

\[ \vec{v}(t), y(t), h(t), \ldots \]

\[ v(t) : \text{velocity of particle at time } t. \]

(rate of change of position wrt. time)

\[ v(t) = \frac{dx}{dt} \]

\[ a(t) : \text{acceleration of particle at time } t \]

(rate of change of velocity with respect to time)
\[ a(t) = \frac{\text{dv}}{\text{dt}} = \frac{\text{d}^2x}{\text{dt}^2} \]

**Note:**

\( v(t) \) answers questions like "how fast is particle moving?" and "in what direction is particle moving?" (Speedometer answers first question.)

\( a(t) \) answers questions like "how quickly is particle speeding up or slowing down?"

**Special case:** Constant acceleration

Freefall near surface of Earth is approximately constant acc.

For gravity, \( a = -g = -9.8 \text{ m/s}^2 \)

If a particle has constant
acceleration, then its position is given by:

\[ x(t) = x_0 + v_0 t + \frac{1}{2} a t^2 \]

- \( x_0 \): initial position
- \( v_0 \): initial velocity
- \( a \): (constant) acceleration

(\text{Newton/Galileo equation})

Q: Does this really have constant acceleration?

A: Verify!

\[ x(t) = x_0 + v_0 t + \frac{1}{2} a t^2 \]

\[ v(t) = v_0 + a t \]

\[ a(t) = a \]

\[ \text{Stone is thrown upwards at } 20 \text{m/s at a height of } 100 \text{m}. \]

\[ \text{What is its peak height and at what?} \]
(a) What is max velocity of stone?
(b) What is max height?
(c) What is impact velocity?
(d) What is total distance traveled by stone?

Use $a = -g = -9.8 \text{ m/s}^2$

Solution:

Since acceleration is constant, the height of stone has form

$$h(t) = h_0 + v_0 t + \frac{1}{2} a t^2$$

$h_0 = 100 \text{ m}$
$v_0 = 20 \text{ m/s}$
$a = -9.8 \text{ m/s}^2$

So the height of the particle at time $t$ is

$$h(t) = 100 + 20t - 4.9t^2$$

(a) The velocity at time $t$ is

$$v(t) = 20 - 9.8t \quad (t \geq 0)$$
So $v_{\text{max}} = 20$

(b) Max height occurs when $v = 0$.

$$0 = v(t) = 20 - 9.8t$$

$$\Rightarrow t = \frac{20}{9.8}$$

when stone reaches max height.

So the max height is

$$h_{\text{max}} = h\left(\frac{20}{9.8}\right) = 100 + 20\left(\frac{20}{9.8}\right) - 4.9\left(\frac{20}{9.8}\right)^2$$

$$h_{\text{max}} = 120.4 \text{ m}$$

(c) "Impact velocity" is velocity of stone as it hits ground.
The stone hits ground when \( h = 0 \).

\[
0 = h(t) = 100 + 20t - 4.9t^2
\]

\[
t = 6.998 \quad \text{when stone hits the ground}
\]

So we have

\[
V_{\text{impact}} = v(6.998) = 20 - 9.8(6.998)
\]

\[
V_{\text{impact}} = -48.58 \text{ m/s}
\]

\( \text{Note: Total distance traveled is not } \Delta h = h(6.998) - h(0) = -100 \text{ m.} \)
Part 1:
\[ d = |120.4 - 100| = 20.4 \text{ m} \]

Part 2:
\[ d = |120.4 - 0| = 120.4 \text{ m} \]

Total distance: \( d_{\text{total}} = 140.8 \text{ m} \)

Ex. 3

The position of a particle along the \( x \)-axis is given by
\[ x(t) = 3t^3 - 40.5t^2 + 162t \]
for \( 0 \leq t \leq 8 \).

(a) When is particle advancing? When is particle retreating?

(b) When does particle have negative acceleration?

(c) What is total distance traveled by particle (from \( t=0 \) to \( t=8 \))?
Solution:

(a) "advancing" $\iff v > 0$
"retreating" $\iff v < 0$
"at rest" $\iff v = 0$

\[ x(t) = 3t^3 - 40.5t^2 + 162t \]
\[ u(t) = 9t^2 - 81t + 162 \]

To solve either "$v > 0$" or "$v < 0$", first solve "$v = 0$".

\[ 0 = u(t) = 9(t-3)(t-6) \]
\[ \implies t = 3 \quad \text{or} \quad t = 6 \]

when particle is at rest

Now calculate sign chart for $u$.

\begin{center}
\begin{tikzpicture}
\draw (-1,0) -- (8,0);
\foreach \x in {0,3,6,8} {\draw[black,fill=white] (\x,.1) circle (2pt);}
\draw (-1,.5) -- (8,.5);
\foreach \x in {0,3,6,8} {\draw[black,fill=black] (\x,.5) circle (2pt);}
\node at (0,.7) {+}; \node at (3,.7) {-}; \node at (6,.7) {+}; \node at (8,.7) {+};
\end{tikzpicture}
\end{center}
\[ v(t) = 9(t-3)(t-6) \]

\( (0,3) : \quad v(1) = 9(-2)(-5) > 0 \)

\( (3,6) : \quad v(4) = 9(1)(-2) < 0 \)

\( (6,8) : \quad v(7) = 9(4)(1) > 0 \)

advancing: \( (0,3) \cup (6,8) \)

retreating: \( (3,6) \)

\( b) \quad v(t) = 9t^2 - 81t + 162 \)

\[ a(t) = 18t - 81 \]

To solve “\( a > 0 \)” or “\( a < 0 \)”, first solve “\( a = 0 \)”.

\[ 0 = a(t) = 18t - 81 \]

\[ \rightarrow t = 4.5 \]

Now calculate sign chart for \( a(t) \).

\[ a(t) = 18(t-4.5) \]
\( (0, 4.5) : a(t) = 18(-3.5) < 0 \)
\( (4.5, 8) : a(t) = 18(5-4.5) > 0 \)

Positive acc.: \( (4.5, 8) \)

Negative acc.: \( (0, 4.5) \)

(c) Particle reverses direction at \( t=3 \) and \( t=6 \).

\[
x(t) = 3t^3 - 40.5t^2 + 162t
\]

Part 1: \( 0 \leq t \leq 3 \)
\[
d_1 = |202.5 - 0| = 202.5
\]

Part 2: \( 3 \leq t \leq 6 \)
\[
d_2 = |162 - 202.5| = 40.5
\]

Part 3: \( 6 \leq t \leq 8 \)
Total distance traveled is
\[ d = d_1 + d_2 + d_3 = 321 \]