Sections 3.2/3.3: Techniques of Differentiation

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>$0$</td>
</tr>
<tr>
<td>$x^n$</td>
<td>$nx^{n-1}$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x$</td>
</tr>
<tr>
<td>$\ln(x)$</td>
<td>$\frac{1}{x}$</td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td>$\cos(x)$</td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td>$-\sin(x)$</td>
</tr>
<tr>
<td>$\tan(x)$</td>
<td>$\sec(x)^2$</td>
</tr>
<tr>
<td>$\sec(x)$</td>
<td>$\sec(x)\tan(x)$</td>
</tr>
<tr>
<td>$\csc(x)$</td>
<td>$-\csc(x)\cot(x)$</td>
</tr>
<tr>
<td>$\cot(x)$</td>
<td>$-\csc(x)^2$</td>
</tr>
</tbody>
</table>

"Power Rule"

This table must be memorized

Advanced rules to combine basic rules:

<table>
<thead>
<tr>
<th>$F(x)$</th>
<th>$F'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f + g$</td>
<td>$f' + g'$</td>
</tr>
<tr>
<td>$cf$</td>
<td>$cf'$</td>
</tr>
<tr>
<td>$fg$</td>
<td>$f'g + fg'$</td>
</tr>
<tr>
<td>$\frac{f}{g}$</td>
<td>$\frac{f'g - fg'}{g^2}$</td>
</tr>
</tbody>
</table>

Product Rule
Quotient Rule
**Definition:**

\[ f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \]

\[ f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

<table>
<thead>
<tr>
<th>Leibniz Notation</th>
<th>Lagrange Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Derivative:</td>
<td>( \frac{df}{dx} )</td>
</tr>
<tr>
<td>2nd Derivative:</td>
<td>( \frac{d^2f}{dx^2} )</td>
</tr>
<tr>
<td>3rd Derivative:</td>
<td>( \frac{d^3f}{dx^2} )</td>
</tr>
<tr>
<td>4th Derivative:</td>
<td>( \frac{d^4f}{dx^4} )</td>
</tr>
<tr>
<td>Nth Derivative:</td>
<td>( \frac{d^nf}{dx^n} )</td>
</tr>
</tbody>
</table>
Ex. 1
Verify that \( \frac{d}{dx} (\tan(x)) = \sec(x)^2 \) using the other rules.

Solution:
We will use Quotient Rule
\[
\frac{d}{dx} (\tan(x)) = \frac{d}{dx} \left( \frac{\sin(x)}{\cos(x)} \right) = \frac{f'}{g} - \frac{f}{g'}
\]
where
\[
f = \sin(x) \\
g = \cos(x)
\]
\[
f' = \cos(x) \\
g' = -\sin(x)
\]
\[
\frac{f'}{g} = \frac{\cos(x)}{\cos(x)} = 1 \\
\frac{f}{g'} = \frac{\sin(x)}{-\sin(x)} = -1
\]
\[
\frac{f'}{g} - \frac{f}{g'} = \frac{1}{\cos(x)} - \frac{-1}{\cos(x)} = \frac{2}{\cos(x)} = 2\sec(x)
\]
\[
\frac{d}{dx} (\tan(x)) = 2\sec(x)
\]

Ex. 2
Calculate \( \frac{d}{dx} \left( \frac{7x^2}{x^3\sqrt{x}} \right) \)

Solution:
We should use Quotient Rule first.
Then we use power rule and Product Rule. But do we?

**General tip**
Always simplify before differentiation.

\[
f(x) = \frac{7x^2}{x^3 \sqrt{x}} = 7x^2 x^{-3} x^{-\frac{1}{2}} = 7x^{-\frac{3}{2}}
\]

Now we don't need Quotient Rule or Product Rule!

\[
f'(x) = \frac{d}{dx} (7x^{-\frac{3}{2}}) = 7 \cdot \frac{d}{dx} \left( x^{-\frac{3}{2}} \right)
\]

\[
= 7 \left( -\frac{3}{2} \right) x^{-\frac{5}{2}} = -\frac{21}{2} x^{-\frac{5}{2}}
\]

---

**Ex. 3**

Calculate \( h'(x) \) if

\[
h(x) = \cos(x) \ln(x)
\]

**Solution:**

\[
h'(x) \neq -\sin(x) \cdot \frac{1}{x}
\]

We must use Product Rule!
No need to simplify!

\[
h(x) = \frac{\cos(x) \ln(x)}{f \quad g}
\]

\[
h'(x) = \frac{(-\sin(x)) (\ln(x))}{f' \quad g} + \frac{(\cos(x)) \left( \frac{1}{x} \right)}{f \quad g'}
\]

Ex. 4

Calculate \( \frac{d}{dx} (mx + b) \), where \( m \) and \( b \) are constants.

Solution:

\[
\frac{d}{dx} (mx + b) = \frac{d}{dx} (mx) + \frac{d}{dx} (b)
\]

\[
= m \quad \frac{d}{dx} (x^1) + \frac{d}{dx} (b)
\]

\[
= 1 \quad x^0 = 1 + 0 = 0
\]

\[
= m \cdot 1 + 0 = m
\]

Does this make sense? What is the slope of \( f(x) = mx + b \)?
Calculate $h'(x)$ if $h(x) = \frac{x^3 - 1}{x^3 + x}$

Solution:

We use Quotient Rule first.

$$h(x) = \frac{f}{g} = \frac{x^3 - 1}{x^3 + x}$$

$$h'(x) = \frac{f'g - fg'}{g^2}$$

$\frac{f'}{g} = 3x^2 \quad g = x^3 + x \quad g' = 3x^2 + 1$

$\boxed{h'(x) = \frac{(3x^2)(x^3 + x) - (x^3 - 1)(3x^2 + 1)}{(x^3 + x)^2}}$

Scratch Work

• $f'(x)$:

$$\frac{d}{dx} (x^3 - 1) = \frac{d}{dx} (x^3) + \frac{d}{dx} (-1)$$

$$= 3x^2 + 0 = 3x^2$$
\[ g'(x) : \]
\[ \frac{d}{dx}(x^3 + x) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x) = 3x^2 + 1 \]

**Ex. 6**

Calculate \( h'(x) \) if

\[ h(x) = \frac{x\sqrt{x} \tan(x)}{e^x - e^3} \]

**Solution:**

\[ h(x) = \frac{x^{3/2} \tan(x)}{e^x - e^3} \]

We use Quotient Rule as main stencil and Product Rule within that.

\[ h'(x) = \frac{\left(\frac{2}{2} \cdot \frac{1}{2} x^{1/2} \tan(x) + x^{3/2} \sec(x)^2 \right) \left( e^x - e^3 \right) - \left( x^{3/2} \tan(x) \right) \left( e^x \right)}{\left( e^x - e^3 \right)^2 \cdot g^2} \]
Scratch Work

- \( f(x) = \frac{x^{3/2}}{\tan(x)} \)

\[
f'(x) = \left( \frac{3}{2} x^{1/2} \right) (\tan(x)) + \left( x^{3/2} \right) (\sec(x)^2)
\]

\[
f'(x) = \frac{3}{2} x^{1/2} \tan(x) + x^{3/2} \sec(x)^2
\]

- \( g(x) = e^x - e^3 \)

\[
g'(x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(e^3) = e^x - 0 = e^x
\]

Which rule do we use?

\[
\frac{d}{dx}(e^3) = \begin{cases} 
3e^2 & \text{Power Rule? No!} \\
e^3 & \text{e^x Rule? No!} \\
0 & \text{Constant Rule? Yes!!} \\
3e^3 & \text{Combination? No!}
\end{cases}
\]

Ex. 7

Find the tangent line to \( f(x) \)
at  $x = 1$.

$$f(x) = x^3 - \frac{3}{x^2}$$

**Solution:**
The slope of the tangent line is $f'(1)$. To find $f'(x)$, first rewrite $f(x)$.

$$f(x) = x^3 - 3x^{-2}$$

$$f'(x) = 3x^2 - 3(-2x^{-3})$$

$$f'(x) = 3x^2 + 6x^{-3}$$

Point: $(1, f(1)) = (1, -2)$

**Equation of tangent line:**

No! $y - (-2) = (3x^2 + 6x^{-3})(x - 1)$

The slope should be a number!

Slope: $f'(1) = 3 + 6 = 9$

**Equation of tangent line:**
Ex. 8

Let \( f(x) = \frac{2x-3}{x+1} \). Find all values of \( a \) for which the tangent line at \( x = a \) is perpendicular to the line \( 3x + 2y = 5 \).

Solution:

What is the question really asking?

The question is asking you to find the
x-coordinates of the two marked points, where the tangent line is perpendicular to the given line $3x + 2y = 5$. Okay, so now let's solve the problem...

The slope of $3x + 2y = 5$ is $m = -\frac{3}{2}$.
(Solve for $y$: $y = -\frac{3}{2}x + \frac{5}{2}$.)

So the slope of the desired tangent line(s) is $\frac{2}{3}$.

(We are equivalently asking, “Where does the tangent line have slope 2/3?”)

So we solve the equation

$$f'(x) = \frac{2}{3}.$$  

So let's set up this equation.

$$f(x) = \frac{2x - 3}{x + 1} \rightarrow F$$

Recall:

$$\frac{d}{dx} (mx + b) = m$$
Find \( f''(x) \) if

\[ f(x) = \pi e^x + \cos(x) - 2 \sqrt{x} \]

Solution:

\[ f(x) = \pi e^x + \cos(x) - 2x^{1/2} \]
1. \( f'(x) = \pi e^x - \sin(x) - 2\left(\frac{1}{2}\right) x^{-\frac{1}{2}} \)
\( f'(x) = \pi e^x - \sin(x) - x^{-\frac{1}{2}} \)

2. \( f''(x) = \pi e^x - \cos(x) + \frac{1}{2} x^{- \frac{3}{2}} \)

3. \( f'''(x) = \pi e^x + \sin(x) + \frac{1}{2} \left(-\frac{3}{2}\right) x^{- \frac{5}{2}} \)
\( f'''(x) = \pi e^x + \sin(x) - \frac{3}{4} x^{- \frac{5}{2}} \)

Ex. 10

Find normal line to \( f(x) \) at \( x = \frac{\pi}{4} \).

\[ f(x) = x^2 \tan(x) \]

(Note: The normal line to \( f \) at \( x = a \) is the line through \((a, f(a))\) and perpendicular to the tangent line.)

Solution:
First we find the slope of the tangent
line using \( f'(x) \). (Product Rule)

\[
f(x) = \frac{x^2 \tan(x)}{F \div G}
\]

\[
f'(x) = \frac{(2x)(\tan(x)) + (x^2)(\sec(x))^2}{F' \div G + F \div G'}
\]

So the slope of the tangent line is

\[
f'\left(\frac{\pi}{4}\right) = \left(\frac{\pi}{2}\right)(1) + \left(\frac{\pi^2}{16}\right)(2)
\]

\[
= \frac{\pi}{2} + \frac{\pi^2}{8} = \frac{4\pi + \pi^2}{8}
\]

The normal line is perpendicular to the tangent line, so the slope of the normal line is:

\[
\text{Slope: } m = -\frac{1}{f'(\frac{\pi}{4})} = -\frac{8}{4\pi + \pi^2}
\]

The normal line passes through \( (\frac{\pi}{4}, f(\frac{\pi}{4})) \):
Point: \( \left( \frac{\pi}{4}, \frac{\pi^2}{16} \right) = \left( \frac{\pi}{4}, \frac{\pi^2}{16} \right) \)

(Recall \( f(x) = x^2 \tan(x) \).)

So now we have:

Equation of normal line:

\[
y - \frac{\pi^2}{16} = \frac{-8}{4\pi + \pi^2} \left( x - \frac{\pi}{4} \right)
\]