Section 3.1: Derivatives and Tangents

The slope of the secant line is:
\[ m = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h} \]

The slope of the tangent line is:
\[ m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \]

Definition: The line tangent to the graph of \( y = f(x) \) at \( x = a \) is...
The derivative function \( f'(a) \) gives slopes of tangent lines.

What is graphical relationship between \( y = f(x) \) and \( y = f'(x) \)?

**Ex. 1**
Given graph of $y = f(x)$, sketch graph of $y = f'(x)$.

**Solution:**

1. Where is $f'(x) = 0$? (At what $x$-values is tangent line horizontal?)
   - Only at $x = 3$. (So $f'(3) = 0$.)
2. Where is $f'(x) > 0$? $(3, \infty)$.
3. Where is $f'(x) < 0$? $(-\infty, 3)$.
Ex. 2

Find tangent line to \( f(x) \) at \( x = 1 \).

\[ f(x) = x^3 + 2x - 1 \]

Solution:

By definition, slope of tangent line is \( f'(1) \). By definition, \( f'(1) \) is the number

\[
f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}
\]

(Put \( a = 1 \) in definition)

\[
= \lim_{h \to 0} \frac{[(1+h)^3 + 2(1+h) - 1] - 2}{h}
\]

\[
(1+h)^3 = (1+h)(1+h)^2
\]

\[
= (1+h)(1+2h+h^2)
\]

\[
= (1+2h+h^2) + h(1+2h+h^2)
\]
\[ = 1 + 3h + 3h^2 + h^3 \]

* Pascal’s Triangle

\[
\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & & & & \\
1 & 2 & 1 & & & \\
1 & 3 & 3 & 1 & & \\
1 & 4 & 6 & 4 & 1 & \\
\end{array}
\]

\[
\lim_{h \to 0} \frac{1 + 3h + 3h^2 + h^3 + 2 + 2h - 1 - 2}{h} = \frac{5h + 3h^2 + h^3}{h}
\]

\[
\lim_{h \to 0} \frac{5h + 3h^2 + h^3}{h} = 5
\]

So slope of tangent line is 5. Line passes through \((1, f(1)) = (1, 2)\). So equation of tangent line is:

\[ y - 2 = 5(x - 1) \]
Ex. 3

Let \( f(x) = \sqrt{x} \). Find the derivative \( f'(x) \) for all \( x \).

Solution:

Clearly, \( f'(x) \) does not exist for \( x < 0 \).

(Any \( x \)-value not in domain of \( f \) cannot be in domain of \( f' \).)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

(Put \( a = x \) in definition)

\[
= \lim_{h \to 0} \left( \frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \quad \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}
\]

\[\text{NO!}\]

\[
= \lim_{h \to 0} \left( \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)
\]

\[
l = (\sqrt{x} + h - \sqrt{x})
\]
\[
\lim_{h \to 0} \left( \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{1}{\sqrt{x+h} + \sqrt{x}} \right)
\]

Now what? We cannot just sub \( h = 0 \). Why? What if \( x = 0 \)?

**Case 1:** \( x = 0 \)

\[
f'(0) = \lim_{h \to 0} \left( \frac{1}{\sqrt{0+h} + \sqrt{0}} \right)
\]

\[
= \lim_{h \to 0} \frac{1}{\sqrt{h}}
\]

This limit does not exist.

**Case 2:** \( x > 0 \)

\[
f'(x) = \lim_{h \to 0} \left( \frac{1}{\sqrt{x+h} + \sqrt{x}} \right)
\]

\[
= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
\]
So the derivative \( f'(x) \) is...

\[
f'(x) = \begin{cases} 
  \text{does not exist} & \text{if } x = 0 \\
  \frac{1}{2\sqrt{x}} & \text{if } x > 0 
\end{cases}
\]

**Ex. 4**

Let \( f(x) = |x| \). Calculate:

(a) \( f'(-3) \)

(b) \( f'(0) \)

**Solution:**

(a) By definition,

\[
f'(-3) = \lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h}
\]

\[
= \lim_{h \to 0} \frac{|-3+h| - 3}{h}
\]

\[
(-3+h)-3 = |-3|+h-3 = 3+h-3 = h
\]

\[
\text{No!}
\]
Recall: \([|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}\]

So what can we say about the expression \(|-3+h|\)?

Since \(h \to 0\), we may assume \(-3+h\) is close to \(-3\). So \(-3+h < 0\). So \(|-3+h| = -(-3+h)\).

\[
= \lim_{h \to 0} \frac{-(-3+h) - 3}{h} = \lim_{h \to 0} \frac{3-h - 3}{h}
\]

\[
= \lim_{h \to 0} \left( \frac{-h}{h} \right) = \lim_{h \to 0} (-1) = -1
\]

So \(f'(-3) = -1\).

(b) By definition,

\[
f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}
\]

\[
= \lim_{h \to 0} \frac{|0+h| - |0|}{h}
\]
Start with same analysis as part (a). What can we say about $|h|$? Since $h$ itself can be negative or positive, we examine one-sided limits.

$$
\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \left( \frac{h}{h} \right) = \lim_{h \to 0^+} (1) = 1
$$

$$
\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \left( \frac{-h}{h} \right) = \lim_{h \to 0^-} (-1) = -1
$$

So $\lim_{h \to 0} \frac{|h|}{h}$ does not exist.

So $f'(0)$ does not exist.

**Terminology**

If $f'(a)$ exists, we say $f$ is differentiable at $x = a$. 
When we compute the derivative of $f$, we **differentiate** $f$. **We do not** “derive” $f$.

How can a function $f$ fail to be differentiable at a point?

**Thm:** If $f$ is differentiable at $x=a$, then $f$ is continuous at $x=a$.

**Ex.**
If $f$ is discontinuous at $x=a$, then $f$ is not differentiable at $x=a$.
If graph of $f$ has kink, corner, or cusp at $x = a$, then $f$ is not differentiable at $x = a$.

*Examples (at $x = 0$)*

$|x|$, $x^{2/3}$  
(cornor), (cusp)

*Ex!*

If $f$ has a **vertical** tangent line at $x = a$, then $f$ is not differentiable at $x = a$. 
Example (at $x = 0$)

\[ x^{1/3} \]

Ex. 5

Find tangent line to $f(x) = \frac{1}{x}$ at $x = 3$.

Solution:

By definition, slope of tangent line is

\[ f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \]

\[ = \lim_{h \to 0} \left( \frac{1}{3+h} - \frac{1}{3} \right) \]
\[ h \to 0 \left( \frac{1}{3+h} - \frac{1}{3} \right) \cdot \frac{3(3+h)}{h} \]

\[ = \lim_{h \to 0} \left( \frac{3 - (3+h)}{3h(3+h)} \right) \]

\[ = \lim_{h \to 0} \left( \frac{-h}{3h(3+h)} \right) = \lim_{h \to 0} \left( \frac{-1}{3(3+h)} \right) \]

\[ = -\frac{1}{9} \]

So tangent line has slope \(-\frac{1}{9}\) and passes through \((3, \frac{1}{3})\). Equation of tangent line is

\[ y - \frac{1}{3} = -\frac{1}{9}(x-3) \]

Ex. 6
Calculate $f'(x)$.

(a) $f(x) = x$
(b) $f(x) = x^2$
(c) $f(x) = x^3$
(d) $f(x) = x^4$

Solution:

(a) $f'(x) = \lim_{h \to 0} \frac{(x+h) - x}{h}$

$= \lim_{h \to 0} \left( \frac{h}{h} \right) = \lim_{h \to 0} (1) = 1 \times 0$

(b) $f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$

$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$

$= \lim_{h \to 0} (2x + h) = 2x^1$

(c) $f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$

$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$
\[ h \to 0 \quad h \]

\[ = \lim_{h \to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \]

\[ = \lim_{h \to 0} \left( 3x^2 + 3hx + h^2 \right) = 3x^2 \]

\[(d) f'(x) = \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} \]

\[= \lim_{h \to 0} \frac{x^4 + 4hx^3 + 6h^2x^2 + 4h^3x + h^4 - x^4}{h} \]

\[= \lim_{h \to 0} \left( 4x^3 + 6hx^2 + 4h^2x + h^3 \right) = 4x^3 \]

**Summary**

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( 1x^0 )</td>
</tr>
<tr>
<td>( x^2 )</td>
<td>( 2x^1 )</td>
</tr>
<tr>
<td>( x^3 )</td>
<td>( 3x^2 )</td>
</tr>
<tr>
<td>( x^4 )</td>
<td>( 4x^3 )</td>
</tr>
</tbody>
</table>
**Power Rule**

If \( f(x) = x^n \), then

\[
f'(x) = n x^{n-1}
\]

We also write this as

\[
\frac{d}{dx} (x^n) = n x^{n-1}
\]