Section 2.1: Intro to Limits

Motivation: average velocity vs. instantaneous velocity

$x(t)$: position of particle
$v(t)$: (instantaneous) velocity of particle.

Average velocity over time interval $[t_1, t_2]$:

$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$

Ex. 1

Suppose $x(t) = 16t^2$. Estimate the instantaneous velocity at $t = 2$.

Solution:
We can estimate the velocity at $t = 2$ with the average velocity.

Average velocity over $[2, 2.1]$:

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x(2.1) - x(2)}{2.1 - 2}$$

$$= \frac{16 \cdot (2.1)^2 - 16 \cdot (2)^2}{0.1} = 65.6$$

If we shrink the length of the
time interval, we should get a better approximation of the instantaneous velocity.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Δt</th>
<th>Average Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>[2, 2.01]</td>
<td>0.01</td>
<td>( \bar{v} = 64.16 )</td>
</tr>
<tr>
<td>[2, 2.001]</td>
<td>0.001</td>
<td>( \bar{v} = 64.016 )</td>
</tr>
<tr>
<td>[1.99, 2]</td>
<td>0.01</td>
<td>( \bar{v} = 63.84 )</td>
</tr>
<tr>
<td>[1.999, 2]</td>
<td>0.001</td>
<td>( \bar{v} = 63.984 )</td>
</tr>
</tbody>
</table>

what is the general relationship between these columns?

"As Δt gets smaller, \( \bar{v} \) gets closer to 64. (and one endpoint was fixed at \( t=2 \))"

We estimate that \( v(2) \approx 64 \).
Our intuition is written precisely as

\[ \lim_{ \Delta t \to 0 } \overline{v} = 64 \]

OR

\[ \lim_{ t \to 2 } \frac{x(t) - x(2)}{t - 2} = 64 \]

Q: How can we calculate \( v(2) \) exactly?

A: Use \( \overline{v} \), but calculate \( \overline{v} \) symbolically for a very tiny time interval.

Let \( h > 0 \) (think of \( h \) as a very tiny number). What is the average velocity over \( [2, 2+h] \).
\[ \bar{V} = \frac{X_{\text{final}} - X_{\text{initial}}}{t_{\text{final}} - t_{\text{initial}}} \]

Recall: 
\[ x(t) = 16t^2 \]

\[ \bar{V} = \frac{X(2+h) - X(2)}{2+h - 2} \]

\[ \bar{V} = \frac{16(2+h)^2 - 16(2)^2}{h} \]

\[ \bar{V} = \frac{16(h^2 + 4 + 4h) - 64}{h} \]

\[ \bar{V} = \frac{16h^2 + 64 + 64h - 64}{h} \]

\[ \bar{V} = \frac{16h^2 + 64h}{h} = \frac{h(16h + 64)}{h} \]

\[ \bar{V} = 64 + 16h \]

This is the average velocity over \([2, 2+h]\), which serves
as an approximation of the instantaneous velocity at \( t=2 \).

If \( h (=\Delta t) \) is close to 0, then \( \overline{v} \) is close to 64!

We write this as

\[
\lim_{h \to 0} \overline{v} = 64
\]

General definition of limit:

\[
\lim_{x \to c} f(x) = L
\]

This means the values of \( f(x) \) can be made arbitrarily close to \( L \) as long as we choose \( x \)-values arbitrarily close to \( c \).
Use a table of values to estimate the value of

$$\lim_{x \to 3} \left( \frac{x^2 - 2x - 3}{x - 3} \right)$$

Solution:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = f(x) = \frac{x^2 - 2x - 3}{x - 3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>3.5</td>
</tr>
<tr>
<td>2.9</td>
<td>3.9</td>
</tr>
<tr>
<td>2.99</td>
<td>3.99</td>
</tr>
<tr>
<td>3</td>
<td>undefined</td>
</tr>
<tr>
<td>3.01</td>
<td>4.01</td>
</tr>
<tr>
<td>3.1</td>
<td>4.1</td>
</tr>
<tr>
<td>3.5</td>
<td>4.5</td>
</tr>
</tbody>
</table>

This table suggests $$\lim_{x \to 3} f(x) = 4.$$
Ex. 3

Use a graph to estimate value of

\[ \lim_{x \to 3} \left( \frac{x^2 - 2x - 3}{x - 3} \right) \]

Solution:

\[ f(x) = \begin{cases} \frac{x^2 - 2x - 3}{x - 3} & \text{if } x \neq 3 \\ \text{undefined} & \text{if } x = 3 \end{cases} \]

Some algebra .... (assume \(x \neq 3\))

\[ \frac{x^2 - 2x - 3}{x - 3} = \frac{(x-3)(x+1)}{x-3} = x+1 \]

In other words,

\[ f(x) = \begin{cases} x + 1 & \text{if } x \neq 3 \\ \text{undefined} & \text{if } x = 3 \end{cases} \]
Graph of $f(x)$:

$y = f(x)$ as $x \rightarrow 3$, what happens to the $y$-coordinates?

$$\lim_{x \rightarrow 3} f(x) = 4$$

\[\text{\rightarrow crucial observation:} \quad \text{``} \lim_{x \rightarrow c} f(x) \text{'' does not care about what is happening at } x = c!\]

**Ex. 4**

Calculate the limit with algebra.

$$\lim_{x \rightarrow 3} \left( \frac{x^2 - 2x - 3}{x - 3} \right)$$
Solution:

\[
\lim_{x \to 3} \left( \frac{x^2 - 2x - 3}{x - 3} \right) = \lim_{x \to 2} \left( \frac{(x-3)(x+1)}{x-3} \right)
\]

\[
= \lim_{x \to 3} (x+1) = 3 + 1 = 4
\]

Is this step okay? Yes! The limit does not care about what is happening at \( x = 3 \), only near \( x = 3 \).

**Note:**

\[
\frac{x^2 - 2x - 3}{x - 3} \neq x + 1 \quad \text{(Why?)}
\]

So "\( \lim \)" tells you that you may assume \( x \neq 3 \).
So far all of the limits we have done are **two-sided limits**.

One-sided limits:

- **Left-sided limits**:
  \[
  \lim_{{x \to c^-}} f(x) = L
  \]

  This means the values of \( f(x) \) can be made arbitrarily close to \( L \) as long as we choose \( x \)-values arbitrarily close to \( c \). … AND \( x < c \).

- **Right-sided limits**:
  \[
  \lim_{{x \to c^+}} f(x) = L
  \]

  This means the values of \( f(x) \)
can be made arbitrarily close to \( L \) as long as we choose \( x \)-values arbitrarily close to \( c \)....

... AND \( x > c \)

**Ex. 5**

Use the graph to calculate each limit.

![Graph of a function](image)

**Solution:**

(a) \( \lim_{x \to -2^-} f(x) = 3 \)  \( \lim_{x \to -2} f(x) = 3 \)

(b) \( \lim_{x \to -2^+} f(x) = 3 \)  \( f(-2) = \)
Note:

\[ \lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L \quad \text{AND} \quad \lim_{x \to c^+} f(x) = L \]

In particular, if the one-sided limits are not equal, the two-sided limit does not exist.

(b) \( \lim_{x \to -1^-} f(x) = -1 \quad \lim_{x \to -1} f(x) \text{ does not exist} \)
\[ \lim_{x \to -1^+} f(x) = 1 \quad f(-1) = -1 \]

\[ \lim_{x \to 1^-} f(x) = 1 \quad \lim_{x \to 1} f(x) \text{ dne.} \]

\[ \lim_{x \to 1^+} f(x) = 3 \quad f(1) = 2 \]
When does $\lim_{x \to c} f(x)$ not exist?

1. One-sided limits not equal.
   - If $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$, then
     $\lim_{x \to c} f(x)$ dne.
   - If at least one one-sided limit dne, then $\lim_{x \to c} f(x)$ dne.

2. Infinite Limits
   - If $\lim_{x \to c} f(x) = \infty$ (or $-\infty$), then technically $\lim_{x \to c} f(x)$ dne.
This is a special case of when $\lim_{x \to c} f(x)$ does not exist. If $x$ gets closer to 0, values of $\frac{1}{x^2}$ get as large as you want. So we use the symbol "\(\infty\)" to denote this.

* We will revisit infinite limits later. Problems will instruct you whether to write "\(\infty\)" or "dne".

3. Infinite Oscillation

$$\lim_{x \to 0} \sin \left( \frac{1}{x} \right) \text{ dne.}$$

* In HW, but beyond scope of course
Section 2.2: Calculating Limits

Direct Substitution Property (DSP)

If \( \lim_{x \to c} f(x) = f(c) \), then \( f \) has the DSP (at \( x = c \)).

What functions have the DSP?

All of the following (or their domain):

1. polynomials
2. rational functions
3. algebraic functions
4. trigonometric
5. exponential (\( e^x, 2^x, \frac{1}{3^x} \), etc.)
6. logarithmic (\( \log_{10}(x), \ln(x) \), ...)

Ex. 1

Calculate \( \lim_{x \to 3} \left( \frac{\sqrt{x^2+1} - 5x^{2/3}}{x^3 - 1} \right)^{1/5} \)

Solution:
Let \( f(x) = \left( \frac{\sqrt{x^2+1} - 5x^{2/3}}{x^3-1} \right)^{1/5} \).

Then \( f \) is an algebraic function. So \( f \) has the DSP at \( x=3 \) since \( x=3 \) is in its domain.

\[
\lim_{x \to 3} \left( \frac{\sqrt{x^2+1} - 5x^{2/3}}{x^3-1} \right)^{1/5} = \left( \frac{\sqrt{10} - 5 \cdot 3^{2/3}}{26} \right)^{1/5}
\]

**Ex. 2**

Calculate \( \lim_{x \to 3} \left( \frac{x^2+x-12}{x-3} \right) \)

**Solution:**

The function \( f(x) = \frac{x^2+x-12}{x-3} \) does not have the DSP at \( x=3 \). So we must use algebra to transform \( f(x) \) into a function with the DSP.
\[ \lim_{x \to 3} \left( \frac{x^2 + x - 12}{x - 3} \right) = \lim_{x \to 3} \left( \frac{(x-3)(x+4)}{x-3} \right) \]

factoring is always okay

\[ \lim_{x \to 3} (x+4) = 3 + 4 = 7 \]

canceling is okay when computing limits because we only care about what happens near \( x = 3 \).

**Ex. 3**

Calculate \( \lim_{x \to 1} \left( \frac{1-\sqrt{x}}{1-x^2} \right) \)

**Solution:**

Note: Direct substitution (DS) of \( x = 1 \) gives \( \frac{0}{0} \), which is undefined. In particular, this does not necessarily
limit does not exist.

2. This indicates there are common factors to cancel out (assuming the function is algebraic.)

\[
\lim_{x \to 1} \left( \frac{1-\sqrt{x}}{1-x^2} \right) = \lim_{x \to 1} \left( \frac{1-\sqrt{x}}{(1-x)(1+x)} \right)
\]

Goal: use algebra to cancel out any factor causing division by zero!

\[
= \lim_{x \to 1} \left( \frac{1-\sqrt{x}}{(1-x)(1+x)} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} \right)
\]

\[
(a-b)(a+b) = a^2 - b^2 \\
(1-\sqrt{x})(1+\sqrt{x}) = 1^2 - (\sqrt{x})^2 = 1 - x
\]

\[
= \lim_{x \to 1} \left( \frac{1-x}{(1-x)(1+x)(1+\sqrt{x})} \right)
\]
\[ \lim_{x \to 1} \frac{1}{(1+x)(1+\sqrt{x})} = \frac{1}{4} \]

**Ex. 4**

Let \( f(x) = \begin{cases} 
  x^2 + 3 & \text{if } x < 1 \\
  10 - x & \text{if } 1 \leq x \leq 2 \\
  6x - x^2 & \text{if } x > 2 
\end{cases} \)

(a) \( \lim_{x \to 1} f(x) \)

(b) \( \lim_{x \to 2} f(x) \)

**Solution:**

When examining piecewise functions
at a transition point, we must often examine the left and right limits separately.

(a) \[ \lim_{{x \to 1^-}} f(x) = \lim_{{x \to 1^-}} (x^2 + 3) = 1 + 3 = 4 \]

This means \( x \) is close to 1 AND \( x < 1 \)

By definition, \( f(x) = x^2 + 3 \) if \( x \) is close to 1 AND \( x < 1 \).

\[ \lim_{{x \to 1^+}} f(x) = \lim_{{x \to 1^+}} (10 - x) = 10 - 1 = 9 \]

This means \( x \) is close to 1 AND \( x > 1 \)

By definition, \( f(x) = 10 - x \) if \( x \) is close to 1 AND \( x > 1 \).

So \( \lim_{{x \to 1}} f(x) \) does not exist.
(b) \[ f(x) = \begin{cases} x^2 + 3 & \text{if } x < 1 \\ 10 - x & \text{if } 1 \leq x \leq 2 \\ 6x - x^2 & \text{if } x > 2 \end{cases} \]

\[ \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (10 - x) = 10 - 2 = 8 \]

\[ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (6x - x^2) = 12 - 4 = 8 \]

So \[ \lim_{x \to 2} f(x) = 8. \]

Ex. 5

Calculate \[ \lim_{x \to 6} \frac{|x-6|}{x-6}. \]

Solution:

Note: Absolute value function does not play well with algebra!

1. \[ |x-6| \neq x-6 \]
2. \[ |x-6| \neq -(x-6) \]
Recall the definition of $|x|:

$$|x| = \begin{cases} 
-x & \text{if } x < 0 \\
x & \text{if } x \geq 0 
\end{cases}$$

So that means ...

$$|x-6| = \begin{cases} 
-(x-6) & \text{if } x-6 < 0 \\
x-6 & \text{if } x-6 \geq 0 
\end{cases}$$

To calculate our limit, we need to look at the left and right limits because $x=6$ is a transition.
point of $|x-6|$. 

- $\lim_{x \to 6^-} \frac{|x-6|}{x-6} = \lim_{x \to 6^-} \frac{-(-6)}{x-6}$
  
  $x$ is close to 6 AND $x-6 < 0$,
  So then $|x-6| = -(x-6)$

  $= \lim_{x \to 6^-} (-1) = -1$

- $\lim_{x \to 6^+} \frac{|x-6|}{x-6} = \lim_{x \to 6^+} \frac{(x-6)}{x-6}$
  
  $x$ is close to 6 AND $x-6 > 0$
  So then $|x-6| = x-6$

  $= \lim_{x \to 6^+} (1) = 1$

So $\lim_{x \to 6} \frac{|x-6|}{x-6}$ does not exist.
Special limits to memorize:
\[
\lim_{{\theta \to 0}} \left( \frac{\sin(\theta)}{\theta} \right) = 1
\]
\[
\lim_{{\theta \to 0}} \left( \frac{\theta}{\sin(\theta)} \right) = 1
\]

Very Common Mistakes:

1. \[ \frac{\sin(2\theta)}{\theta} \neq \frac{2\sin(\theta)}{\theta} \]

2. " \[ \frac{\sin(\theta)}{\theta} = 1 \]" (why is this wrong?)

Ex. 6

Calculate \[ \lim_{{x \to 0}} \frac{\sin(2x)}{2x} \]

Solution:

False solution:
\[
\lim_{{x \to 0}} \frac{\sin(2x)}{2x} = \lim_{{x \to 0}} \frac{2\sin(x)}{2x} = \lim_{{x \to 0}} \frac{\sin(x)}{x} = 1
\]
The limit
\[ \lim_{x \to 0} \frac{\sin(2x)}{2x} \]
looks a lot like
\[ \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \]

But in what sense? “2x” is playing the role of “\(\theta\).

(1) If \(\theta = 2x\), then
\[
\frac{\sin(2x)}{2x} = \frac{\sin(\theta)}{\theta}
\]

(2) If \(\theta = 2x\), then “\(x \to 0\)” is equivalent to “\(\theta \to 0\)” (Why?)

So we may conclude that
\[ \lim_{x \to 0} \frac{\sin(2x)}{2x} = 1 \]
Similarly . . . .

\[ \lim_{x \to 0} \frac{\sin (3x)}{3x} = 1 \quad (\text{Let } \theta = 3x) \]

\[ \lim_{x \to 0} \frac{\sin (-2x)}{-2x} = 1 \quad (\text{Let } \theta = -2x) \]

\[ \lim_{x \to 0} \frac{\sin \left( \frac{\pi}{\ln(2)} x \right)}{\frac{\pi}{\ln(2)} x} = 1 \quad (\text{Let } \theta = \frac{\pi}{\ln(2)} x) \]

\[ \lim_{x \to 0} \frac{\sin (ax)}{ax} = 1 \quad (a \neq 0) \]

\[ \lim_{x \to 0} \frac{ax}{\sin (ax)} = 1 \quad (a \neq 0) \]

**Ex. 7**

Calculate \( \lim_{x \to 0} \frac{\sin (7x)}{x} \).

**Solution:**

This almost looks like our . . . .
special limit. So use algebra!

\[
\lim_{x \to 0} \left( \frac{\sin(7x)}{x} \right) = \lim_{x \to 0} \left( \frac{\sin(7x)}{7x} \cdot 7 \right)
\]

\[
= \lim_{x \to 0} \left( \frac{\sin(7x)}{7x} \right) \cdot \lim_{x \to 0} (7)
\]

limit = 1 \quad \text{limit} = 7

= 1 \cdot 7 = 7

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**Ex. 8**

Calculate \( \lim_{x \to 0} \frac{\tan(8x)}{\sin(3x)} \).

**Solution:**

\[
\lim_{x \to 0} \left( \frac{\tan(8x)}{\sin(3x)} \right) = \lim_{x \to 0} \left( \frac{\sin(8x)}{\cos(8x) \sin(3x)} \right)
\]

\[
= \lim_{x \to 0} \left( \frac{\sin(8x)}{1} \cdot \frac{1}{\sin(3x)} \cdot \frac{1}{\cos(8x)} \right)
\]
\[
\lim_{x \to 0} \left( \frac{\sin(8x)}{8x} \cdot \frac{3x}{\sin(3x)} \cdot \frac{1}{\cos(8x)} \cdot \frac{8x}{3x} \right)
\]

\[
= \lim_{x \to 0} \left( \frac{\sin(8x)}{8x} \cdot \frac{3x}{\sin(3x)} \cdot \frac{1}{\cos(8x)} \cdot \frac{8}{3} \right)
\]

\[
\text{limit} = 1 \quad \text{limit} = 1 \quad \text{limit} = 1 \quad \text{limit} = \frac{8}{3}
\]

(Use special limit) (use DSP) (use DSP)

\[
= 1 \cdot 1 \cdot 1 \cdot \frac{8}{3} = \frac{8}{3}
\]

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Ex. 9

Calculate \( \lim_{x \to 0} \frac{(x+3)^2 - 9}{x} \)

Solution:

\[
\lim_{x \to 0} \left( \frac{(x+3)^2 - 9}{x} \right) = \lim_{x \to 0} \left( \frac{x^2 + 6x + 9 - 9}{x} \right)
\]

\[
= \lim_{x \to 0} \left( \frac{x(x+6)}{x} \right) = \lim_{x \to 0} (x+6) = 0+6 = 6
\]
Ex. 10

Calculate \( \lim_{x \to 1} \left( \frac{\frac{1}{x} - 1}{x - 1} \right) \)

Solution.

\[
\lim_{x \to 1} \left( \frac{\frac{1}{x} - 1}{x - 1} \right) = \lim_{x \to 1} \left( \frac{\frac{1}{x} - 1}{x - 1} \cdot \frac{x}{x} \right)
\]

\[
= \lim_{x \to 1} \left( \frac{1 - x}{x(x - 1)} \right) = \lim_{x \to 1} \left( \frac{-1}{x(x - 1)} \right)
\]

\[
= \lim_{x \to 1} \left( -\frac{1}{x} \right) = -\frac{1}{1} = -1
\]

Ex. 11

Let \( f(x) = \begin{cases} \frac{x^2 - x}{x - 1} & \text{if } x > 1 \\ \sqrt{1-x} & \text{if } x \leq 1 \end{cases} \)

Calculate \( \lim_{x \to 1} f(x) \)
Solution:

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left( \sqrt{1-x} \right) = \sqrt{1-1} = 0 \]

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left( \frac{x^2 - x}{x - 1} \right) = \lim_{x \to 1^+} \left( \frac{x(x-1)}{x-1} \right) = \lim_{x \to 1^+} (x) = 1 \]

So \( \lim_{x \to 1} f(x) \) does not exist.
Section 2.3: Continuity

**Def:** We say \( f \) is **continuous** at \( x = c \) if

\[
\lim_{{x \to c}} f(x) = f(c)
\]

(i.e., \( f \) has the DSP at \( x = c \).) Otherwise we say \( f \) is **discontinuous** at \( x = c \).

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How can a function fail to be continuous at \( x = c \)? Four types of discontinuity:

1. ![Removable Discontinuity](image)
   - \( \lim_{{x \to c}} f(x) \) exists
   - \( f(c) \) undefined

   OR
2. \( f(c) \) defined but \( \lim_{x \to c} f(x) = f(c) \)

- \( \lim_{x \to c^-} f(x) \) and \( \lim_{x \to c^+} f(x) \) exist but are not equal.

3. \( f(x) \) is infinite as \( x \to c \) or \( c \) is infinite (\( \infty \) or \( -\infty \)).

4. Essential
f is discontinuous at $x = c$, but not (1), (2), or (3).

Ex. 1

Let $f(x) = \frac{x^2 + x - 12}{x - 3}$.

(a) Where is $f$ continuous?
(b) At each value of $x$ where $f$ is discontinuous, can we redefine the value of $f$ so that $f$ is continuous there?

Solution:

(a) For what values of $x$ does $f$ have the DSP? For all $x$ except $x = 3$. So $f$ is continuous on $(-\infty, 3) \cup (3, \infty)$.
(b) Right now $f(3)$ is undefined.
Can we choose a value for \( f(3) \) to make \( f \) continuous at \( x = 3 \)?

**Note:** If \( f \) were continuous at \( x = 3 \), we would have

\[
\lim_{x \to 3} f(x) = f(3)
\]

Left-hand side tells us what the new value of \( f(3) \) should be.

So we compute \( \lim_{x \to 3} f(x) \):

\[
\lim_{x \to 3} \left( \frac{x^2 + x - 12}{x - 3} \right) = \lim_{x \to 3} \left( \frac{(x+4)(x-3)}{x-3} \right)
\]

\[
= \lim_{x \to 3} (x + 4) = 3 + 4 = 7.
\]

So if we define \( f(3) \) to be 7, then \( f \) is continuous at \( x = 3 \).
Ex. 2

Let \( f(x) = \begin{cases} x^2 + 3 & \text{if } x < 0 \\ x - 5 & \text{if } x \geq 0 \end{cases} \)

(a) Where is \( f \) continuous?

(b) At each value of \( x \) where \( f \) is discontinuous, can we redefine the value of \( f \) so that \( f \) is continuous there?

Solution:

(a) Where does \( f \) have the DSP?

Each “piece” \( f \) has the DSP, so the only value of \( x \) for which \( f \) might not have the DSP is \( x = 0 \) (the transition point).

To check whether \( f \) is continuous at \( x = 0 \), we must check whether
$\lim_{x \to 0} f(x) = f(0)$

So the left limit, right limit, and function value should all be equal.

- $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + 3) = 0 + 3 = 3$
- $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x - 5) = 0 - 5 = -5$
- $f(0) = (x - 5)\bigg|_{x=0} = -5$

Since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$, $f$ is discontinuous at $x = 0$. In summary, $f$ is continuous on $(-\infty, 0) \cup (0, \infty)$.

(b) Can we redefine $f(0)$ so that $f$ is continuous at $x = 0$?
Note: The only choice for \( f(0) \) is

\[ f(0) = \lim_{x \to 0} f(x). \]

Since \( \lim_{x \to 0} f(x) \) does not exist, it is impossible to redefine \( f(0) \) to make \( f \) continuous at \( x = 0 \).

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**Ex. 3**

Let

\[ f(x) = \begin{cases} 
3x + 2 & \text{if } x \leq 1 \\
5 & \text{if } 1 < x \leq 3 \\
3x^2 - 1 & \text{if } x > 3
\end{cases} \]

Where is \( f \) continuous?

Solution:

Each “piece” of \( f \) is continuous (Why? In this case, each piece is a polynomial.) But \( f \) may have a discontinuity at \( x = 1 \) or \( x = 3 \).
Continuity at \( x = 1 \)

We must check whether the following numbers are equal:

\[
\lim_{x \to 1^-} f(x), \quad \lim_{x \to 1^+} f(x), \quad f(1)
\]

1. \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x + 2) = 3 + 2 = 5 \)
2. \( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (5) = 5 \)
3. \( f(1) = (3x + 2) \bigg|_{x=1} = 5 \)

So \( f \) is continuous at \( x = 1 \).

Continuity at \( x = 3 \)

1. \( \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (5) = 5 \)
2. \( \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (3x^2 - 1) = 26 \)
\[ f(3) = (5) \bigg|_{x=3} = 5 \]

This last line is not required since the left and right limits already show \( f \) is not continuous.

So \( f \) is discontinuous at \( x = 3 \).

In summary, \( f \) is continuous on \((-\infty, 3) \cup (3, \infty)\).

---

**Ex. 4**

Let \( f(x) = \begin{cases}  
  x + a & \text{if } x < 0 \\
  5 & \text{if } x = 0 \\
  \sin(bx) \div x & \text{if } x > 0
\end{cases} \)

Find the values of \( a \) and \( b \) which would make \( f \) continuous at \( x = 0 \), or show that no such
Solution:
If \( f \) is to be continuous at \( x=0 \), we must have

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)
\]

In principle, these numbers depend on \( a \) and \( b \).

1. \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x+a) = 0+a = a \)

2. \( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left( \frac{\sin (bx)}{x} \right) = \lim_{x \to 0^+} \left( \frac{\sin (bx)}{bx} \cdot b \right) = \lim_{x \to 0^+} \left( \frac{\sin (bx)}{bx} \right) \cdot \lim_{x \to 0^+} (b) = 1 \cdot b = b \)

3. \( f(0) = 5 \)
So we must choose $a$ and $b$ so that $a = b = 5$. (So $a = 5$ and $b = 5$.)

**Intermediate Value Theorem (IVT)**

**Thm:** Suppose $f$ is continuous on the interval $[a, b]$. Then given any number $d$ between $f(a)$ and $f(b)$, there exists a number $c$ in $(a, b)$ such that $f(c) = d$.

What does this mean graphically?
Intuition: Continuous functions are not allowed to "skip" y-values.

Challenge: Given \((a, f(a))\) and \((b, f(b))\), can we draw the graph of \(y = f(x)\) so that some y-values between \(f(a)\) and \(f(b)\) are skipped?

Note: graph of \(f(x)\) never intersects the horizontal line \(y = d\).

This challenge is easy ....

.... as long as \(f\) is allowed to be discontinuous!
The IVT says that if $f$ must be continuous, then this challenge is impossible!

**Proving “$f(x) = 0$” Has a Solution:**

Suppose we have the following:

1. $f$ is continuous on $[a, b]$.
2. $f(a)$ and $f(b)$ have opposite signs (one value is positive, the other negative)

Q: What can you say about the equation “$f(x) = 0$”? 

![Graph showing a function with $f(a)$ and $f(b)$ opposite in sign, indicating a solution at $x = d$.]
A: We conclude that "$f(x)=0$" has at least one solution between $x=a$ and $x=b$.

Ex. 5

Prove that the equation

$$\cos(x) = x^3 - x$$

has a solution in the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

Solution:

Equivalently, we show that

$$\cos(x) - x^3 + x = 0$$

has a solution. Let

$$f(x) = \cos(x) - x^3 + x$$

Now observe the following:

1. $f$ is continuous on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ (Why? $\cos(x)$, $-x^3$, and $x$ are all...
continuous; f(x) is a sum of continuous functions)

\[ f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \left(\frac{\pi}{4}\right)^3 + \frac{\pi}{4} \approx 1.008 \]

\[ f\left(\frac{\pi}{2}\right) = 0 - \left(\frac{\pi}{2}\right)^3 + \frac{\pi}{2} \approx -2.305 \]

(So \( f\left(\frac{\pi}{4}\right) \) and \( f\left(\frac{\pi}{2}\right) \) have opposite signs.)

So by the IVT, the equation \( f(x) = 0 \) has a solution in \( \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \).
Section 2.4: Exponential and Logarithmic Functions

How do we define exponents?

\[ 2^3 = 2 \cdot 2 \cdot 2 \quad \text{(repeated multiplication)} \]

\[ 2^3 \cdot 2^5 = 2^{3+5} = 2^8 \]

\[ a^x \cdot a^y = a^{x+y} \]

\[ 2^{-1} = \frac{1}{2} \]

\[ 2^{-1} \cdot 2^1 = 2^{-1+1} = 2^0 = 1 \]

\[ 2^{-1} = \frac{1}{2^1} \]

\[ 2^{\frac{1}{3}} = y \iff y^3 = 2 \]

\[ 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} = 2^{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 2^1 = 2 \]

\[ (2^{\frac{1}{3}})^3 = 2 \]

This is done to preserve the rule \( a^x \cdot a^y = a^{x+y} \)
\[ 2^{\frac{5}{3}} = (2^{\frac{1}{3}})^5 \quad (a^x a^y = a^{x+y}) \]

\[ 2^{\pi} = ?? \]

We define this using limits and continuity.

**Ex:** How do we define \( b^{\sqrt{2}} \)?

\[
\begin{array}{c|c}
 x & b^x \\
\hline
 1 & b' = b \\
 1.4 & b^{1.4} = b^{14/10} = (b^{1/10})^{14} \\
 1.41 & b^{1.41} = (b^{1/100})^{141} \\
 1.412 & b^{1.412} = (b^{1/1000})^{1412} \\
\end{array}
\]

Limit of these numbers is \( \sqrt{2} \)

We can prove that the limit of these numbers exists

\( b^{\sqrt{2}} \)
In other words, \( b^x \) is \textit{defined} so that \( f(x) = b^x \) is a \textit{continuous function}.

\[
\lim_{x \to c} b^x = b^c
\]

**Properties of Exponential Functions**

\[ y = b^x \]

\((0 < b < 1)\]

Ex: \( \left(\frac{1}{2}\right)^x, e^{-x} \)

\[ y = b^x \]

\((b > 1)\]

Ex: \( 2^x, e^x \)

Algebraic/Calculus Properties:
\cdot a^x a^y = a^{x+y} \quad \cdot a^{-1} = \frac{1}{a}

\cdot (a^x)^y = a^{xy} \quad \cdot a^{\frac{1}{n}} = \sqrt[n]{a}

\cdot \text{domain of } f(x) = b^x: \quad (-\infty, \infty)

\cdot \text{range of } f(x) = b^x: \quad (0, \infty)

\cdot f(x) = b^x \text{ is continuous}

\cdot f(x) = b^x \text{ is monotonic}

\leftrightarrow \text{always increasing } (b > 1)

\text{OR always decreasing } (0 < b < 1)

\underline{\text{Since } f(x) = b^x \text{ is monotonic, } f(x) \text{ is also one-to-one.}}

\leftrightarrow \text{graph passes the horizontal line test}

\underline{\text{So } f(x) = b^x \text{ has an inverse}}
function, which we call the logarithm with base $b$.

$$y = \log_b(x) \iff b^y = x$$

$$y = f^{-1}(x) \iff f(y) = x$$

$$\log_b(b^x) = x \quad \text{(for all } x)$$

$$b^{\log_b(x)} = x \quad \text{(for } x > 0)$$

**Properties of Logarithmic Functions**

Algebraic / Calculus Properties:
• \[ \log_b(x) + \log_b(y) = \log_b(xy) \]
• \[ \log_b(x) - \log_b(y) = \log_b \left( \frac{x}{y} \right) \]
• \[ \log_b(x^y) = y \log_b(x) \]
• Domain of \( f(x) = \log_b(x) \): \( (0, \infty) \)
• Range of \( f(x) = \log_b(x) \): \( (-\infty, \infty) \)
• \( f(x) = \log_b(x) \) is \underline{continuous}.
• \( f(x) = \log_b(x) \) is \underline{monotonic}.

\[ \text{Special Limits with Exp/Log:} \]

\[ y = b^x \]

\[ (0 < b < 1) \quad \text{and} \quad (b > 1) \]
\[ \lim_{x \to -\infty} b^x = \infty \quad \lim_{x \to -\infty} b^x = 0 \]
\[ \lim_{x \to \infty} b^x = 0 \quad \lim_{x \to \infty} b^x = \infty \]

\[ \lim_{x \to 0^+} \log_b x = \infty \quad \lim_{x \to 0^+} \log_b x = -\infty \]
\[ \lim_{x \to \infty} \log_b x = -\infty \quad \lim_{x \to \infty} \log_b x = \infty \]

\[ y = \log_b (x) \]

The number \( e \)

The primary reason we will work (almost) exclusively with base
Chapter 3

Working only with base e is not a big deal.

\[ a^x = e^{x \ln(a)}; \quad \log_a(x) = \frac{\log_e(x)}{\log_e(a)} \]

So how do we define e?

**Note:** Simply stating \( e = 2.71828 \ldots \) is not a definition.

There are several definitions, depending on author and course.

1. \( e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \)
2. \( e \) is the unique number so that
\[
\frac{d}{dx}(e^x) = e^x ; \quad \text{or} \quad \lim_{h \to 0} \frac{e^h - 1}{h} = 1
\]

3. \( e \) is the value of the limit
\[
\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e \quad \text{we use this definition}
\]

**Special Notation**

\[
\log_e(x) = \ln(x)
\]

\[
\exp(x) = e^x
\]

\( \leftarrow \) this is used for typographical reasons: \( e^{x^2 \cos(3x)} = \exp(x^2 \cos(3x)) \).

**Application:** Continuously Compounded Interest
Where does this formula come from?

\[ P(t) = P_0 e^{rt} \]

Let's look at a more general situation. Suppose we invest \( P_0 \) dollars at an APR of \( r \), compounded \( N \) times per year.

After each compounding, the value of the investment grows by the factor

\[ (1 + \frac{r}{N}) \]

So what is value of investment after one year (\( N \) compounding)

\[ A_1 = P_0 \left(1 + \frac{r}{N} \right) \] (after 1 comp.)
\[ A_2 = P_0 \left( 1 + \frac{r}{N} \right) \left( 1 + \frac{r}{N} \right) \]
\[ = P_0 \left( 1 + \frac{r}{N} \right)^2 \quad \text{(after 2 comp.)} \]
\[ A_3 = P_0 \left( 1 + \frac{r}{N} \right)^2 \left( 1 + \frac{r}{N} \right) \]
\[ = P_0 \left( 1 + \frac{r}{N} \right)^3 \quad \text{(after 3 comp.)} \]
\[ \vdots \]
\[ A_N = P_0 \left( 1 + \frac{r}{N} \right)^N \quad \text{(after N comp.)} \]

Q: What happens as \( N \to \infty \)?

A: We call this continuously compounded interest. We can show that
\[ \lim_{N \to \infty} \left( 1 + \frac{r}{N} \right)^N = e^r \] similar to definition of \( e \)

We will show how to compute this later.

\[ \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e \]
So growth factor after one year is \( e^r \). So growth factor after \( t \) years is \( (e^r)^t = e^{rt} \).

**Ex. 1**

Find all solutions to
\[
\ln \left( \frac{x^2}{1-x} \right) = \ln (x) + \ln \left( \frac{2x}{1+x} \right)
\]

**Solution:**

**Note:** \( e^{\ln(x)} = x \) (for \( x > 0 \))

**Note:** \( e^{\ln(a) + \ln(b)} \neq a + b \)

**Better:** \( \ln(a) + \ln(b) = \ln(ab) \)
\[
e^{\ln(a) + \ln(b)} = e^{\ln(ab)} = ab
\]

So first we simplify the RHS of the equation
\[
\ln \left( \frac{x^2}{1-x} \right) = \ln (x) + \ln \left( \frac{2x}{1+x} \right)
\]
(use \( \ln(a) + \ln(b) = \ln(ab) \))

\[
\ln \left( \frac{x^2}{1-x} \right) = \ln \left( \frac{2x^2}{1+x} \right)
\]

Now we can exponentiate both sides with base \( e \) to "cancel" the \( \ln \) function.

\[
\frac{x^2}{1-x} = \frac{2x^2}{1+x}
\]

\[
x^2 (1+x) = 2x^2 (1-x)
\]

\[
x^2 + x^3 = 2x^2 - 2x^3
\]

\[
3x^3 - x^2 = 0
\]

\[
x^2 (3x - 1) = 0
\]

\[
x^2 = 0 \quad \text{OR} \quad 3x - 1 = 0
\]

\[
x = 0 \quad \text{OR} \quad x = \frac{1}{3}
\]

There is a problem! We must
verify the candidate solutions. Why?

\[ \ln \left( \frac{x^2}{1-x} \right) = \ln (x) + \ln \left( \frac{2x}{1+x} \right) \]

\( x = 0 \): No! \( \ln (0) \) is undefined

\( x = \frac{1}{3} \): Okay!
Section 3.1: Derivatives and Tangents

Slope of secant line: \( m = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h} \)

Slope of tangent line: \( m = \lim_{{h \to 0}} \frac{f(a+h) - f(a)}{h} \)

Def: The line tangent to the graph of \( y = f(x) \) at \( x = a \) is the line...
That passes through \((a, f(a))\) with slope

\[ M = \lim_{{h \to 0}} \frac{f(a+h) - f(a)}{h} \]

**Def:** Given \(f(x)\), the number

\[ f'(a) = \lim_{{h \to 0}} \frac{f(a+h) - f(a)}{h} \]

is the **derivative** of \(f\) at \(x = a\).

**Note:** The derivative function \(f'(x)\) gives as its output the value of the slope of the tangent line at \(x\) to \(y = f(x)\).

**What is the graphical relationship between \(f\) and \(f'\)?**

**Ex. 1**

Given graph of \(y = f(x)\), sketch the graph of \(y = f'(x)\).
Solution:

\[ f'(x) < 0 \quad f'(x) > 0 \quad f'(x) < 0 \]

\[ y = f(x) \quad y = f'(x) \]

① Where is \( f'(x) = 0 \)?

← At what \( x \)-values is the slope of the tangent line equal to 0? (horizontal line)
This means \((2,0), (4,0), (6,0),\) and \((8,0)\) are all on the graph of \(f'(x)\). Why? Because
\[
f'(2) = f'(4) = f'(6) = f'(8) = 0
\]

\(\textcircled{2} \) Where is \(f'(x) > 0?\)

\(\textcircled{3} \) Where is \(f'(x) < 0?\)

What is relationship between these points?\[
y = f(x) \quad y = f'(x)
\]
The y-coordinate of pink point must give slope of tangent line at orange point!

**Ex. 2**

Find tangent line to \( f(x) \) at \( x = 1 \).

\[ f(x) = x^3 + 2x - 1 \]

**Solution:**

In the earlier language, \( a = 1 \).

**Point:** \((1, f(1)) = (1, 2)\)

**Slope:** By definition, \( f'(1) \) is the slope.

\[
 f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \quad \text{definition}
\]

\[
 = \lim_{h \to 0} \frac{[(1+h)^3 + 2(1+h) - 1] - 2}{h}
\]
\[
\lim_{h \to 0} \frac{[1 + 3h + 3h^2 + h^3 + 2 + 2h] - 1}{h} = \lim_{h \to 0} \frac{5h + 3h^2 + h^3}{h} = \lim_{h \to 0} \frac{h(5 + 3h + h^2)}{h} = \lim_{h \to 0} (5 + 3h + h^2) = 5 + 0 + 0 = 5
\]

Slope of tangent line

Equation of tangent line:
\[
y - 2 = 5(x - 1)
\]

**Ex. 3**

Let \( f(x) = \sqrt{x} \). Calculate \( f'(x) \).

Solution:

Clearly \( f'(x) \) DNE if \( x < 0 \).

By definition, for \( x > 0 \),

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]
\[
\lim_{h \to 0} \left( \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)
\]
\[
= \lim_{h \to 0} \left( \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \right)
\]
\[
= \lim_{h \to 0} \left( \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \right)
\]
\[
= \lim_{h \to 0} \left( \frac{1}{\sqrt{x+h} + \sqrt{x}} \right)
\]

Now we substitute \( h = 0 \), but not so fast! (What if \( x = 0 \)?)

**Case I: \( x = 0 \)**

\[
f'(0) = \lim_{h \to 0} \left( \frac{1}{\sqrt{0+h} + \sqrt{0}} \right) = \lim_{h \to 0} \frac{1}{\sqrt{h}}
\]

\( \implies \) This limit does not exist.

• Left-hand limit is meaningless
since $\sqrt{h}$ is defined only for $h > 0$.

- Substitution of $h = 0$ gives $1/0$ and there is no cancellation.

So $f'(0)$ DNE.

**Case II: $x > 0$**

$$f'(x) = \left( \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \right) = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

In summary, if $f(x) = \sqrt{x}$,

$$f'(x) = \begin{cases} 
\text{DNE} & \text{if } x \leq 0 \\
\frac{1}{2\sqrt{x}} & \text{if } x > 0 
\end{cases}$$

**Ex. 4**

Let $f(x) = |x|$. Calculate:

(a) $f'(-3)$
(b) $f'(0)$

**Solution:**

(a) By definition,
\[ f(-3) = \lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h} \]

\[ = \lim_{h \to 0} \frac{|-3+h| - 3}{h} \]

**Note:** \(|-3+h| - 3 = (-3) + |h| - 3 \]

WRONG!! \[ = 3 + |h| - 3 \]

\[ = |h| = h \]

Recall \[ |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} \]

So how should we “simplify” \(|-3+h|\)?

Is \(-3+h\) positive or negative?

(We assume \(h\) is close to 0 since we are considering \( \lim_{h \to 0} \).)

If \(h\) is close to 0, then \(-3+h\) is negative. So \(|-3+h| = -(3+h)\).

Back to our limit...
\[
\lim_{h \to 0} \frac{|-3+h| - 3}{h} = \lim_{h \to 0} \frac{-(-3+h) - 3}{h} \\
= \lim_{h \to 0} \frac{3 - h - 3}{h} = \lim_{h \to 0} \left( \frac{-h}{h} \right) = \lim_{h \to 0} (-1) = -1
\]

So \( f'(-3) = -1 \).

(b) By definition,

\[
f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\
= \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}
\]

Since \( h \) can be positive or negative, \(|h|\) is ambiguous. So we must look at left and right limits.

\[
\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1 \\
\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} (1) = 1
\]

\( \Rightarrow h < 0 \)
So $f'(0)$ does not exist.

**Terminology**

- If $f'(a)$ exists, we say $f$ is differentiable at $x=a$.
- When we compute the derivative $f'(x)$, we are differentiating $f$. (We do not “derive” $f$.)

How can a function $f$ fail to be differentiable at a point?
If \( f \) is not continuous at \( x = a \), then \( f \) is not differentiable at \( x = a \).

- A function can be continuous, yet still not differentiable.

* For now, you should be able to recognize such points of non-differentiability by a graph.
* We will revisit functions that are not differentiable in Chapter 4.

Note: If you are interested in more analytic details, see the end of these notes for a preview!
Ex. 5

Find tangent line to \( f(x) = \frac{1}{x} \) at \( x=3 \).

Solution:

**Point:** \((3, f(3)) = (3, \frac{1}{3})\)

**Slope:** \( f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \)

\[
= \lim_{h \to 0} \left( \frac{1}{3+h} - \frac{1}{3} \right) \cdot \frac{3(3+h)}{h}
\]

\[
= \lim_{h \to 0} \left( \frac{3 - (3+h)}{3h(3+h)} \right) = \lim_{h \to 0} \left( \frac{3 - 3 - h}{3h(3+h)} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{-h}{3h(3+h)} \right) = \lim_{h \to 0} \left( \frac{-1}{3(3+h)} \right)
\]

\[
= -\frac{1}{9} \quad \text{slope of tangent line}
\]

**Equation of tangent line:** \( y - \frac{1}{3} = -\frac{1}{9}(x-3) \)
Calculate \( f'(x) \).

(a) \( f(x) = x \)

(b) \( f(x) = x^2 \)

(c) \( f(x) = x^3 \)

(d) \( f(x) = x^4 \)

**Solution:**

(a) \( f'(x) = \lim_{h \to 0} \frac{(x+h) - x}{h} \)

\[ = \lim_{h \to 0} \left( \frac{h}{h} \right) = \lim_{h \to 0} (1) = 1 \]

(b) \( f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \)

\[ = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \]

\[ = \lim_{h \to 0} (2x + h) = 2x \]
(c) \( f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \)

\[ = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \]

\[ = \lim_{h \to 0} \left(3x^2 + 3xh + h^2\right) = 3x^2 \]

(d) \( f'(x) = \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} \)

\[ = \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \]

\[ = \lim_{h \to 0} \left(4x^3 + 6x^2h + 4xh^2 + h^3\right) = 4x^3 \]

**Summary**

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) )</td>
<td>( 1 )</td>
<td>( 2 )</td>
<td>( 3 )</td>
<td>( 4 )</td>
</tr>
</tbody>
</table>

Is there a pattern here? Yes!
Power Rule

If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

We also write this as

$$\frac{d}{dx} (x^n) = nx^{n-1}$$
What types of functions are continuous but not differentiable?

1. Absolute Value: \( f(x) = |x| \)

   The function \( f(x) \) is not differentiable at \( x = 0 \). So \( h(x) = |g(x)| \) is (possibly) not differentiable wherever \( g(x) = 0 \).

   \[ \text{Ex: } h(x) = |x^2 - 4| \text{ is not differentiable at } x = -2 \text{ and } x = 2 \ (x^2 - 4 = 0 \implies x = \pm 2) \]

2. Power Functions: \( f(x) = x^n \ (0 < n < 1) \)

   The function \( f(x) \) is not differentiable at \( x = 0 \). So \( h(x) = g(x)^n \) is (possibly) not differentiable wherever \( g(x) = 0 \).

   \[ \text{Ex: } h(x) = (x^2 - 4)^{2/5} \text{ is not differentiable at } x = -2 \text{ and } x = 2 \ (x^2 - 4 = 0 \implies x = \pm 2) \]

3. Piecewise-Defined Functions (assumed or proven continuous)

   A piecewise-defined functions sometimes fails to be differentiable at the transition points.
### Techniques of Differentiation

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>$0$</td>
</tr>
<tr>
<td>$x^n$</td>
<td>$nx^{n-1}$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x$</td>
</tr>
<tr>
<td>$\ln(x)$</td>
<td>$\frac{1}{x}$</td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td>$\cos(x)$</td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td>$-\sin(x)$</td>
</tr>
<tr>
<td>$\tan(x)$</td>
<td>$\sec(x)^2$</td>
</tr>
<tr>
<td>$\sec(x)$</td>
<td>$\sec(x)\tan(x)$</td>
</tr>
<tr>
<td>$\csc(x)$</td>
<td>$-\csc(x)\cot(x)$</td>
</tr>
<tr>
<td>$\cot(x)$</td>
<td>$-\csc^2(x)$</td>
</tr>
</tbody>
</table>

- "Power Rule"
- This table must be memorized

### Advanced rules to combine basic rules:

<table>
<thead>
<tr>
<th>$F(x)$</th>
<th>$F'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f + g$</td>
<td>$f' + g'$</td>
</tr>
<tr>
<td>$cf$</td>
<td>$cf'$</td>
</tr>
<tr>
<td>$fg$</td>
<td>$f'g + fg'$</td>
</tr>
<tr>
<td>$\frac{f}{g}$</td>
<td>$\frac{f'g - fg'}{g^2}$</td>
</tr>
</tbody>
</table>
# Notation for Derivatives

## Definition:

\[
 f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

\[
 f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

<table>
<thead>
<tr>
<th>Derivative</th>
<th>Leibniz Notation</th>
<th>Lagrange Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Derivative</td>
<td>( \frac{df}{dx} )</td>
<td>( f'(x) )</td>
</tr>
<tr>
<td>2nd Derivative</td>
<td>( \frac{d^2f}{dx^2} )</td>
<td>( f''(x) )</td>
</tr>
<tr>
<td>3rd Derivative</td>
<td>( \frac{d^3f}{dx^3} )</td>
<td>( f'''(x) )</td>
</tr>
<tr>
<td>4th Derivative</td>
<td>( \frac{d^4f}{dx^4} )</td>
<td>( f^{(4)}(x) )</td>
</tr>
<tr>
<td>Nth Derivative</td>
<td>( \frac{d^n f}{dx^n} )</td>
<td>( f^{(n)}(x) )</td>
</tr>
</tbody>
</table>
Ex. 1
Verify that \( \frac{d}{dx} (\tan(x)) = \sec(x)^2 \) using the other rules.

Solution:
We will use Quotient Rule.

\[
\frac{d}{dx} (\tan(x)) = \frac{d}{dx} \left( \frac{\sin(x)}{\cos(x)} \right) \rightarrow f
\]

\[
= \frac{f' \cdot g - f \cdot g'}{g^2}
\]

\[
= \left( \cos(x) \right)^2 \frac{\cos(x)}{\cos(x)^2}
\]

\[
= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2} = \frac{1}{\cos(x)^2} = \sec(x)^2
\]

Ex. 2
Calculate \( \frac{d}{dx} \left( \frac{7x^2}{x^3 \sqrt{x}} \right) \)

Solution:
We should use Quotient Rule first.
Then we use power rule and Product Rule. But do we?

**General tip**

Always simplify before differentiation.

\[ f(x) = \frac{7x^2}{x^3 \sqrt{x}} = 7x^2 x^{-3} x^{-\frac{1}{2}} = 7x^{-\frac{3}{2}} \]

Now we don’t need Quotient Rule or Product Rule!

\[
\begin{align*}
  f'(x) &= \frac{d}{dx} \left( 7x^{-\frac{3}{2}} \right) = 7 \cdot \frac{d}{dx} \left( x^{-\frac{3}{2}} \right) \\
  &= 7 \left( -\frac{3}{2} \right) x^{-\frac{3}{2}} = -\frac{21}{2} x^{-\frac{5}{2}}
\end{align*}
\]

**Ex. 3**

Calculate \( h'(x) \) if

\[ h(x) = \cos(x) \ln(x) \]

**Solution:**

\[ h'(x) \neq -\sin(x) \cdot \frac{1}{x} \quad \text{No!} \]

We must use Product Rule!
\[ h(x) = \frac{\cos(x)}{f} \ln(g) \]

No need to simplify!

\[ h'(x) = \left( -\sin(x) \right) \left( \ln(x) \right) + \cos(x) \left( \frac{1}{x} \right) \]

\[ \frac{f'}{f} \cdot g + \frac{g'}{g} \]

Ex. 4

Calculate \( \frac{d}{dx} (mx + b) \), where \( m \) and \( b \) are constants.

Solution:

\[ \frac{d}{dx} (mx + b) = \frac{d}{dx} (mx) + \frac{d}{dx} (b) \]

\[ = m \left( x^1 \right) + 0 \]

\[ = mx^0 = 1 \quad = 0 \]

\[ = m \cdot 1 + 0 = m \]

Does this make sense? What is the slope of \( f(x) = mx + b \)?
Ex. 5

Calculate \( h'(x) \) if

\[
h(x) = \frac{x^3 - 1}{x^3 + x}
\]

**Solution:**
We use Quotient Rule first.

\[
h(x) = \frac{f}{g} \quad \Rightarrow \quad f(x) = \frac{x^3 - 1}{x^3 + x} \quad \text{and} \quad g(x) = x^3 + x
\]

\[
h'(x) = \frac{f'g - fg'}{g^2} = \frac{(3x^2)(x^3 + x) - (x^3 - 1)(3x^2 + 1)}{(x^3 + x)^2}
\]

**Scratch Work**

- \( f'(x) \):

\[
\frac{d}{dx} (x^3 - 1) = \frac{d}{dx} (x^3) + \frac{d}{dx} (-1)
\]

\[
= 3x^2 + 0 = 3x^2
\]
\[ g'(x) = \frac{d}{dx} (x^3 + x) = \frac{d}{dx} (x^3) + \frac{d}{dx} (x) = 3x^2 + 1 \]

**Ex. 6**

Calculate \( h'(x) \) if

\[ h(x) = \frac{x \sqrt{x} \tan(x)}{e^x - e^3} \]

**Solution:**

\[ h(x) = \frac{x^{3/2} \tan(x)}{e^x - e^3} \]

We use **Quotient Rule** as main stencil and **Product Rule** within that.

\[ h'(x) = \frac{\frac{3}{2} x^{1/2} \tan(x) + x^{3/2} \sec^2(x)}{2(e^x - e^3) - (x^{3/2} \tan(x))}{e^x}} \]

\[ = \frac{(e^x - e^3)^2}{g^2} \]
Scratch Work

1. **f(x) = \( \frac{x^{3/2}}{F} \tan (x) \)**

   \[ f'(x) = \left( \frac{3}{2} x^{1/2} \right) \left( \tan (x) \right) + \left( x^{3/2} \right) \left( \sec(x)^2 \right) \]

   \[ \frac{f'}{G} + \frac{F}{G'} \]

   \[ f'(x) = \frac{3}{2} x^{1/2} \tan (x) + x^{3/2} \sec (x)^2 \]

2. **g(x) = e^x - e^3**

   \[ g'(x) = \frac{d}{dx} (e^x) - \frac{d}{dx} (e^3) = e^x - 0 = e^x \]

   **Which rule do we use?**

   \[ \begin{cases} 
   3e^2 \text{ ?? Power Rule? No!} \\
   e^3 \text{ ?? e^x Rule? No!} \\
   0 \text{ ?? Constant Rule? Yes!!} \\
   3e^3 \text{ ?? Combination? No!} 
   \end{cases} \]

---

**Ex. 7**

Find the tangent line to f(x)
at \( x = 1 \).

\[
f(x) = x^3 - \frac{3}{x^2}
\]

**Solution:**

The slope of the tangent line is \( f'(1) \). To find \( f'(x) \), first rewrite \( f(x) \).

\[
f(x) = x^3 - 3\cdot x^{-2}
\]

\[
f'(x) = 3x^2 - 3 \cdot (-2x^{-3})
\]

\[
f'(x) = 3x^2 + 6x^{-3}
\]

Point: \( (1, f(1)) = (1, -2) \)

**Equation of tangent line:**

\[
y - (-2) = (3x^2 + 6x^{-3})(x - 1)
\]

No! The slope should be a number!

Slope: \( f'(1) = 3 + 6 = 9 \)

**Equation of tangent line:**
Let \( f(x) = \frac{2x - 3}{x + 1} \). Find all values of \( a \) for which the tangent line at \( x = a \) is perpendicular to the line \( 3x + 2y = 5 \).

**Solution:**

What is the question really asking?

The question is asking you to find the
x-coordinates of the two marked points, where the tangent line is perpendicular to the given line $3x + 2y = 5$. Okay, so now let's solve the problem...

The slope of $3x + 2y = 5$ is $m = -\frac{3}{2}$. (Solve for $y$: $y = -\frac{3}{2}x + \frac{5}{2}$.)

So the slope of the desired tangent line(s) is $\frac{2}{3}$.

(We are equivalently asking, "Where does the tangent line have slope $\frac{2}{3}$?"

So we solve the equation

\[ f'(x) = \frac{2}{3}. \]

So let's set up this equation.

\[ f(x) = \frac{2x - 3}{x + 1} \]

[Recall: \( \frac{d}{dx} (mx + b) = m \)]
\[
f'(x) = \frac{F'}{G'} \left( \frac{F}{G} \right) (x + 1) - \frac{F}{G} (2x - 3) \left( \frac{1}{x + 1} \right)
\]
\[
= \frac{2(x + 1) - (2x - 3)}{(x + 1)^2} = \frac{5}{(x + 1)^2}
\]

Now solve \( f'(x) = \frac{2}{3} \).

\[
\frac{5}{(x + 1)^2} = \frac{2}{3} \implies \frac{15}{2} = (x + 1)^2
\]

\[
x = -1 \pm \sqrt{\frac{15}{2}}
\]

\[
x = -1 + \sqrt{\frac{15}{2}} \quad \text{OR} \quad x = -1 - \sqrt{\frac{15}{2}}
\]

**Ex. 9**

Find \( f''(x) \) if

\[
f(x) = \pi e^x + \cos(x) - 2 \sqrt{x}
\]

**Solution:**

\[
f(x) = \pi e^x + \cos(x) - 2x^{1/2}
\]
\[ f'(x) = \pi e^x - \sin(x) - 2\left(\frac{1}{2}\right) x^{-1/2} \]
\[ f'(x) = \pi e^x - \sin(x) - x^{-1/2} \]
\[ f''(x) = \pi e^x - \cos(x) + \frac{1}{2} x^{-3/2} \]
\[ f'''(x) = \pi e^x + \sin(x) + \frac{1}{2} \left(\frac{-3}{2}\right) x^{-5/2} \]

\[ f'''(x) = \pi e^x + \sin(x) - \frac{3}{4} x^{-5/2} \]

**Ex. 10**

Find normal line to \( f(x) \) at \( x = \pi/4 \).

\[ f(x) = x^2 \tan(x) \]

(Note: The normal line to \( f \) at \( x = a \) is the line through \((a, f(a))\) and perpendicular to the tangent line.)

**Solution:**

First we find the slope of the tangent
line using \( f'(x) \). (Product Rule)

\[
f(x) = \frac{x^2 \tan(x)}{F \ G}
\]

\[
f'(x) = \frac{(2x)(\tan(x)) + (x^2)(\sec(x)^2)}{F' \ G + F \ G'}
\]

So the slope of the tangent line is

\[
f'(\frac{\pi}{4}) = \left(\frac{\pi}{2}\right)(1) + \left(\frac{\pi^2}{16}\right)(2)
\]

\[
= \frac{\pi}{2} + \frac{\pi^2}{8} = \frac{4\pi + \pi^2}{8}
\]

The normal line is perpendicular to the tangent line, so the slope of the normal line is:

Slope: \( m = -\frac{1}{f'(\pi/4)} = -\frac{8}{4\pi + \pi^2} \)

The normal line passes through \((\frac{\pi}{4}, f(\frac{\pi}{4}))\):
Point: \( \left( \frac{\pi}{4}, \frac{\pi^2}{16} \cdot 1 \right) = \left( \frac{\pi}{4}, \frac{\pi^2}{16} \right) \)

(Recall \( f(x) = x^2 \tan(x) \).)

So now we have:

Equation of normal line:

\[
y - \frac{\pi^2}{16} = -\frac{8}{4\pi + \pi^2} \left( x - \frac{\pi}{4} \right)
\]
Section 3.4: Rectilinear Motion

Basic Definitions

\[ x(t) : \text{position along x-axis at time } t \]

or \( s(t), y(t), h(t), \text{etc.} \)

\[ v(t) : \text{velocity at time } t \]

\[ v(t) = \frac{dx}{dt} \]

\[ a(t) : \text{acceleration at time } t \]

\[ a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} \]

Special Case: Constant Acceleration

Freefall near surface of Earth is approximately constant

If a particle has constant acceleration \( a \), then its position has the form

\[ x(t) = x_0 + v_0 t + \frac{1}{2} at^2 \] "Galileo equation"

\[ x_0 : \text{initial position, } x(0) \]
Vo: initial velocity, $v(0)$

$a$: (constant) acceleration

For gravity, $a = -g = -9.8 \text{ m/s}^2$ (Earth)

Q: Does $x(t)$ above really give rise to constant acceleration?

A: $x(t) = x_0 + v_0 t + \frac{1}{2} at^2$

$\rightarrow x(0) = x_0 + 0 + 0 = x_0$

$m(t) = v(t) = v_0 + at$

$\rightarrow v(0) = v_0 + 0 = v_0$

$a(t) = a$ (constant!)

---

**Ex. 1**

Stone is thrown upward at 20 m/s at a height of 100 m.

(a) What is max velocity of stone?

(b) What is max height?

(c) What is impact velocity?

(d) What is total distance traveled by
Stone until impact? 
Use \( a = -g = -9.8 \text{ m/s}^2 \).

**Solution:**
Since the acceleration is constant, the height of stone has the form:

\[
h(t) = h_0 + v_0 t + \frac{1}{2} a t^2
\]

\[
h_0 = 100
\]
\[
v_0 = 20
\]
\[
a = -9.8
\]

\[
h(t) = 100 + 20t - 4.9t^2
\]

(a) The velocity is:

\[
v(t) = h'(t) = 20 - 9.8t
\]

Since \( v(t) \) is decreasing on \([0, \infty)\), the max velocity is \( v(0) = 20 \).

(b) To find max height, solve \( h'(t) = 0 \).
\[ h'(t) = 20 - 9.8t = 0 \implies t = \frac{20}{9.8} \]

Time when particle reaches maximum height.

Observe that \( h''(t) = -9.8 \) < 0 for all \( t \). So \( h \) is concave down on \((0, \infty)\). So \( t = \frac{20}{9.8} \) gives max height.

\[ h_{\text{max}} = h\left(\frac{20}{9.8}\right) = 120.4 \text{ m} \]

\[ h(t) = 100 + 20t - 4.9t^2 \]

(c) The impact velocity is the velocity of stone just as it hits the ground. First find when stone hits the ground.

\[ 0 = 100 + 20t - 4.9t^2 \]

Use quadratic formula.

\[ t = -2.926 \quad \text{or} \quad t = 6.998 \]

Time when stone hits ground.

So the impact velocity is
\[ V_{\text{impact}} = V(6.998) = \Theta 48.58 \]
\[ V(t) = 20 - 9.8t \quad \text{stone is falling down} \]

**Note:** The total distance traveled is not simply \( \Delta h = h(6.998) - h(0) = -100 \) and also not \( |\Delta h| = 100 \).

**Part 1** from \( t=0 \) to \( t=\frac{20}{9.8} \):
\[ d_1 = \int h(\frac{20}{9.8}) - h(0) \, dt = 20.4 \]

**Part 2** from \( t=20/9.8 \) to \( t=6.998 \):
\[ d_2 = \int h(6.998) - h(20/9.8) \, dt = \int 0 - 120.4 \, dt = 120.4 \]

\[ d_{\text{total}} = d_1 + d_2 = 140.8 \, \text{m} \]
The position of a particle along x-axis is given by

\[ x(t) = 3t^3 - 40.5t^2 + 62t \]

for \( 0 \leq t \leq 8 \).

(a) When is particle at rest? \( (v = 0) \)
When is particle advancing? \( (v > 0) \)
When is particle retreating? \( (v < 0) \)

(b) What is the total distance traveled by the particle? (from \( t=0 \) to \( t=8 \))

Solution:

(a) The velocity of particle is

\[ v(t) = 9t^2 - 81t + 162 \]

First solve \( v(t) = 0 \).

\[ 0 = 9(t-3)(t-6) \implies t = 3, \ t = 6 \]

Construct a sign chart for \( v(t) \) to solve both "\( v(t) > 0 \)" and "\( v(t) < 0 \)"
\[
\begin{align*}
\text{sign of } v(t) & \text{ test point} \\
v(t) &= 9(t-3)(t-6) \\
v(1) &= \text{sign} = \text{--} \\
v(4) &= \text{sign} = \text{--} \\
v(7) &= \text{sign} = \text{++} \\
\text{advancing: } [0,3), (6,8] \quad (v>0) \\
\text{retreating: } (3,6) \quad (v<0)
\end{align*}
\]

(b) Note: The total distance traveled is not \( \Delta x = x(8) - x(0) \) or \( |\Delta x| \).

The particle reverses direction at both \( t=3 \) and \( t=6 \). So we divide the path of the particle into three pieces.
\[ x(t) = 3t^3 - 40.5t^2 + 162t \]

\[ \text{Part 1} \quad t = 0 \text{ to } t = 3 \]
\[ d_1 = |x(3) - x(0)| = |202.5 - 0| = 202.5 \]

\[ \text{Part 2} \quad t = 3 \text{ to } t = 6 \]
\[ d_2 = |x(6) - x(3)| = |162 - 202.5| = 40.5 \]

\[ \text{Part 3} \quad t = 6 \text{ to } t = 8 \]
\[ d_3 = |x(8) - x(6)| = |240 - 162| = 78 \]

\[ d_{\text{total}} = d_1 + d_2 + d_3 = 321 \]
Section 3.5: The Chain Rule

How do we differentiate ....

\[ f(x) = \sin(x) \quad \frac{d}{dx} \sin(x) = \cos(x) \]

\[ f(x) = 2 \sin(x) \quad \frac{d}{dx} (cf) = c \frac{df}{dx} \]

Product Rule

But what about a composition?

\[ f(x) = \sin(2x) \]

Chain Rule

**Thm. (Chain Rule)**

If \( f \) and \( g \) are differentiable, then

\[ \frac{d}{dx} \left[ f \left( g(x) \right) \right] = f'(g(x)) \cdot g'(x) \]

- derivative of outside function evaluated at inside function derivative of inside function

**Ex. 1**

Calculate \( \frac{d}{dx} \left( \sin \left( x^2 \right) \right) \).

**Solution:**
Outside: \( f(x) = \sin(x) \)

Inside: \( g(x) = x^2 \)

\[ h(x) = \sin(x^2) \]

\[ h'(x) = \cos(x^2) \cdot 2x \]

---

**Ex. 2**

Calculate \( \frac{d}{dx} (e^{3x+4}) \)

**Solution:**

Outside: \( e^x \)

Inside: \( 3x+4 \) (exponent)

\[ h(x) = e^{3x+4} \]

\[ h'(x) = e^{3x+4} \cdot 3 \left( \frac{d}{dx}(mx+b) = m \right) \]

---

**Ex. 3**

Calculate \( \frac{d}{dx} \left( e^{\tan(x)} \right) \).

**Solution:**

Outside: \( e^x \)

Inside: \( \tan(x) \) (exponent)
\[ h(x) = e^{\tan(x)} \]
\[ h'(x) = e^{\tan(x)} \cdot \sec(x)^2 \cdot \frac{d}{dx} \tan(x) \]

**Ex. 4**

Calculate \( \frac{d}{dx} \left( \ln \left( x^3 + x \right) \right) \).

**Solution:**

Outside: \( \ln \left( x \right) \)

Inside: \( x^3 + x \)

\[ h(x) = \ln \left( x^3 + x \right) \]
\[ h'(x) = \frac{1}{x^3 + x} \cdot (3x^2 + 1) = \frac{3x^2 + 1}{x^3 + x} \]

Chain rule

\[ \frac{d}{dx} \left( x^3 + x \right) = 3x^2 + 1 \]

**Ex. 5**

Calculate \( \frac{d}{dx} \left( \sin(e^x) \cos(x) \right) \).

**Solution:**
Use product rule first.

\[ h(x) = \frac{\sin(e^x) \cdot \cos(x)}{F \cdot G} \]

\[ h'(x) = \frac{(\cos(e^x) \cdot e^x)(\cos x) + (\sin e^x)(-\sin x)}{F' \cdot G + F \cdot G'} \]

**Scratch Work**

- \( F(x) = \sin(e^x) \)
  - **Outside**: \( f(x) = \sin(x) \)
  - **Inside**: \( g(x) = e^x \)

\[ F'(x) = \cos(e^x) \cdot e^x \]

**Ex. 6**

Calculate \( \frac{d}{dx} \left( \sqrt{\frac{x^3}{1-x}} \right) \).

**Solution:**

First rewrite as \( h(x) = \left( \frac{x^3}{1-x} \right)^{1/2} \)
Our main stencil is chain rule with power rule.

Outside: \( x^{1/2} \)

Inside: \( \frac{x^3}{1-x} \) (we will need quotient rule for inside term)

\[
h'(x) = \frac{1}{2} \left( \frac{x^3}{1-x} \right)^{-1/2} \cdot \frac{(1-x) \cdot 3x^2 - x^3 \cdot (-1)}{(1-x)^2}
\]

\[
\text{derivative of inside function}
\]

Note: The stencil you use is very sensitive to how you write \( h(x) \).

\[
h(x) = \left( \frac{x^3}{1-x} \right)^{1/2} = \frac{x^{3/2}}{(1-x)^{1/2}} = x^{3/2} \cdot (1-x)^{-1/2}
\]

(Chain Rule with Power Rule) (Quotient Rule) (Product Rule)

Ex. 7

Calculate \( \frac{d}{dx} \left( \ln \left( \tan \left( e^x \right) \right) \right) \).
Solution:

Note that \( h(x) = \ln (\tan (e^x)) \) is naturally a composition of 3 functions. So we need 2 applications of chain rule.

Outside: \( \ln (x) \)

Inside: \( \tan (e^x) \) (Chain rule for the inside term.)

\[
h(x) = \ln (\tan (e^x))
\]

\[
h'(x) = \frac{1}{\tan (e^x)} \cdot \frac{d}{dx} (\tan (e^x))
\]

This term comes from first application of chain rule.

For second chain rule \( \Rightarrow \)

Outside: \( \tan (x) \)

Inside: \( e^x \)

\[
h'(x) = \frac{1}{\tan (e^x)} \cdot \sec^2(e^x) \cdot e^x
\]
Ex. 8

Calculate \( \frac{d}{dx} \left( \sqrt[3]{\cos \left( (7x-5)^4 \right)} \right) \)

Solution:

\[ h(x) = \left[ \cos \left( (7x-5)^4 \right) \right]^{1/3} \]

We need 3 applications of chain rule. (We should end up with 4 separate factors.)

\[ h'(x) = \ldots \text{ (peel off each layer like an onion)} \]

\[ = \frac{1}{3} \left( \cos \left( (7x-5)^4 \right) \right)^{-2/3} \cdot \left( -\sin \left( (7x-5)^4 \right) \right) \cdot 4 \cdot (7x-5)^3 \cdot 7 \]

---

Ex. 9

Let \( f(x) = x \sqrt{1-3x} \). Find the \( x \)-coordinate of each point where the line tangent to the graph of \( y=f(x) \) is horizontal.
Solution:
Since \( f'(x) \) gives us the slopes of the tangent lines and a horizontal line has slope zero, we must solve the equation “\( f'(x) = 0 \).”

\[
f(x) = x \left(1 - 3x\right)^{\frac{1}{2}}
\]

(Use product rule as main stencil)

\[
f'(x) = \frac{1 \cdot \left(1 - 3x\right)^{\frac{1}{2}} + x \cdot \frac{1}{2} \left(1 - 3x\right)^{-\frac{1}{2}} \cdot (-3)}{F' \quad G + F \quad G'}
\]

* For \( G'(x) \) we need chain rule.

Now solve “\( f'(x) = 0 \)” for \( x \).

\[
\left(1 - 3x\right)^{\frac{1}{2}} - \frac{3}{2} x \left(1 - 3x\right)^{-\frac{1}{2}} = 0
\]

(Multiply both sides by \( (1-3x)^{\frac{1}{2}} \).)

\[
(1 - 3x)^{1} - \frac{3}{2} x (1 - 3x)^{0} = 0
\]
\[ 1 - 3x - \frac{3}{2}x = 0 \]
\[ x = \frac{2}{9} \]

Check: Is \( x = \frac{2}{9} \) in the domain of \( f(x) = x (1 - 3x)^{1/2} \)? Yes! Because \( 1 - 3 \left( \frac{2}{9} \right) \geq 0 \).

**Ex. 10**

Selected values of \( f, f', g, \) and \( g' \) are given in the table below:

<table>
<thead>
<tr>
<th></th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( g(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
<td>-4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>-3</td>
<td>2</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>-4</td>
<td>3</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

(a) Let \( F(x) = \frac{f(x)}{g(x)} \). Find \( F'(0) \).

(b) Let \( G(x) = f(xg(x)) \). Find \( G'(1) \).

**Solution:**
(a) Use quotient rule to find $F'(x)$.

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$F'(o) = \frac{f'(o)g(o) - f(o)g'(o)}{g(o)^2}$$

$$F'(o) = \frac{(-4)(4) - (-1)(2)}{(4)^2} = \frac{-7}{8}$$

(b) Our main stencil for $G(x)$ is chain rule.

Outside: $f(x)$

Inside: $xg(x)$ (use product rule)

$$G'(x) = f'(xg(x)) \cdot \frac{d}{dx}(xg(x))$$

This term comes from chain rule. Now use product rule.

$$G'(x) = f'(xg(x)) \cdot \left[1 \cdot g(x) + x \cdot g'(x)\right]$$
Now substitute \( x = 1 \).

\[ G'(1) = f'(1 \cdot g(1)) \left[ 1 \cdot g(1) + 1 \cdot g'(1) \right] \]

\[ G'(1) = f'(2) \left[ 2 + (-4) \right] \]

\[ G'(1) = 3 \cdot [2-4] = -6 \]

---

**Ex. 11**

My exam #1: Fall 2018

Calculate \( f'(x) \) where \( f \) is the function below.

\[ f(x) = \left( \frac{x^8 \sin(3x)}{\ln(x) - \ln(11)} \right)^{2/3} \]

After calculating the derivative, do not simplify your answer.

**Solution:**

\[ f(x) = \left( \frac{x^8 \sin(3x)}{\ln(x) - \ln(11)} \right)^{2/3} \]

- Chain Rule + Power Rule
- Quotient Rule
- Product Rule \((x^8 \sin(3x))\)
The final answer is:

\[ f'(x) = \frac{2}{3} \left( \frac{x^{\frac{8}{3}} \sin 3x}{\ln x - \ln 11} \right)^{-\frac{1}{3}} \cdot \frac{(\ln x - \ln 11)(8x^7 \sin 3x + 3x^8 \cos 3x) - (x^{\frac{8}{3}} \sin 3x) \cdot \frac{1}{x}}{(\ln x - \ln 11)^2} \]
Section 3.6: Implicit Differentiation

Sometimes $x$ and $y$ are related by an equation but we cannot solve for one as a function of the other.

$$x^2 + y^2 = 1$$

- Top half: $y = \sqrt{1-x^2}$
- Bottom half: $y = -\sqrt{1-x^2}$

The more general situation is not nearly as nice:
We say that $y$ is a function of $x$ “locally.” You have generally no hope of finding a formula for that function! But you can find the derivative of local functions.

**Ex. 1**

Suppose $x^2 + y^2 = 1$. Find the tangent line to the graph at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$.

**Solution:**

We make no attempt to solve for $y$ in terms of $x$.

But $y = f(x)$ locally!

\[
x^2 + f(x)^2 = 1 \quad \text{(1)}
\]

The slope of the tangent line is $\frac{dy}{dx}$, or $f'(x)$. We can differentiate
both sides of (1) with respect to 

x to get another equation that 
is true for all points on the 
curve.

REMEMBER CHAIN RULE!

\[ 2x + 2f(x)f'(x) = 0 \]  \hspace{1cm} (2)

Scratch Work:

- \( F(x) = [f(x)]^2 \) (Power + Chain)
  - Outside: \( x^2 \)
  - Inside: \( f(x) \)

\[
\frac{d}{dx} \left( [f(x)]^2 \right) = 2f(x) \cdot f'(x)
\]

We will typically write (2) as

\[ 2x + 2y \cdot \frac{dy}{dx} = 0 \]  \hspace{1cm} (2)*

Now solve algebraically for \( \frac{dy}{dx} \)
\[ x + y \frac{dy}{dx} = 0 \]

\[
\frac{dy}{dx} = -\frac{x}{y}
\]

(For implicit functions, \( \frac{dy}{dx} \) is allowed to depend explicitly on both \( x \) and \( y \).)

The slope of the tangent line at \((\frac{1}{2}, \frac{\sqrt{3}}{2})\) is

\[
\left. \frac{dy}{dx} \right|_{x=\frac{1}{2}, \ y=\frac{\sqrt{3}}{2}} = -\frac{\frac{1}{2}}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}
\]

Substitute \( x=\frac{1}{2} \) and \( y=\frac{\sqrt{3}}{2} \) into \( \frac{dy}{dx} = -\frac{x}{y} \).

**Point:** \((\frac{1}{2}, \frac{\sqrt{3}}{2})\)

**Slope:** \(-\frac{1}{\sqrt{3}}\)

**Equation:**

\[
y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}} \left( x - \frac{1}{2} \right)
\]
Ex. 2

Suppose $x$ and $y$ are implicitly related by the equation

$$x^2 + 3y^2 + xy = 10$$

Find $\frac{dy}{dx}$ for points on the curve.

**Solution:**

Use implicit differentiation

(Differentiate w.r.t. $x$)

$$x^2 + 3y^2 + xy = 10$$

$$2x + 6y \frac{dy}{dx} + y + x \frac{dy}{dx} = 0$$

**Scratch Work:**

- $3y^2$

$$\frac{d}{dx} (3y^2) = 6y \cdot \frac{dy}{dx}$$

- $3f(x)^2$

$$\frac{d}{dx} (3f(x)^2) = 3 \cdot 2f(x) \cdot f'(x)$$

$y$ is in disguise, a function of $x$. 
Calculus is done! Now use algebra to solve for \( \frac{dy}{dx} \).

\[
2x + 6y \frac{dy}{dx} + y + x \frac{dy}{dx} = 0
\]

(Algebraic structure of the equation is **linear** in \( \frac{dy}{dx} \)!)
Ex. 3

Find an equation of the tangent line to the graph of

\[ x^3 + y^3 = 3xy \]

at \( \left( \frac{2}{3}, \frac{4}{3} \right) \).

Solution:
Use implicit differentiation to find \( \frac{dy}{dx} \), which gives the slope of the tangent line.

Differentiate \( \text{wrt. } x \)

\[ x^3 + y^3 = 3xy \]
\[ 3x^2 + 3y^2 \frac{dy}{dx} = 3\left(x \frac{dy}{dx} + 1 \cdot y\right) \]

**Scratch Work:**

\[ y^3 \]

\[ \frac{d}{dx} \left(y^3\right) = 3y^2 \cdot \frac{dy}{dx} \]

\[ \frac{d}{dx} \left(\left[f(x)\right]^3\right) = 3 \left(f(x)\right)^2 \cdot f'(x) \]

\[ 3x^2 + 3y^2 \frac{dy}{dx} = 3x \frac{dy}{dx} + 3y \quad (1) \]

Now we find the slope of the tangent line. There are two ways to proceed.

1. **Solve for** \( \frac{dy}{dx} \), **THEN**
   Substitute \( x = \frac{2}{3}, \ y = \frac{4}{3} \).
2. **Substitute** \( x = \frac{2}{3}, \ y = \frac{4}{3} \), **THEN**
   Solve for \( \frac{dy}{dx} \).
Method 2 has easier algebra. We substitute \( x = \frac{2}{3} \) and \( y = \frac{4}{3} \) into equation (1).

\[
\frac{4}{3} + \frac{1b}{3} \cdot \frac{dy}{dx} = 2 \frac{dy}{dx} + 4
\]

Now solve for \( \frac{dy}{dx} \).

\[
4 + 16 \frac{dy}{dx} = 6 \frac{dy}{dx} + 12
\]

\[
10 \frac{dy}{dx} = 8
\]

\[
\frac{dy}{dx} = \frac{4}{5}
\]

The slope of the tangent line at the point \( \left( \frac{2}{3}, \frac{4}{3} \right) \).

The equation of the tangent line:

\[
y - \frac{4}{3} = \frac{4}{5} \left( x - \frac{2}{3} \right)
\]

Ex. 4

Let \( f(x) = x^x \). Find \( f'(x) \).
Solution:

- Power Rule?
  \[
  \frac{d}{dx} (x^x) = x \cdot x^{x-1} = x^x
  \]
  No! Cannot use Power Rule because exponent is not constant.

- Exponential Rule?
  \[
  \frac{d}{dx} (a^x) = a^x \cdot \ln(a)
  \]
  \[
  \frac{d}{dx} (x^x) = x^x \cdot \ln(x)
  \]
  No! Cannot use Exponential Rule because base is not constant.

None of our rules have \( x \) in both base and exponent. We need to be clever.

**Method 1**  Rewrite \( f(x) \), then use usual rules for derivatives
\[ f(x) = x^x \]
\[ f(x) = e^{\ln(x) \cdot x} \]
\[ f(x) = e^{x \ln(x)} \]

Now use chain rule!

Outside: \( e^x \)

Inside: \( x \ln(x) \) (Product Rule)

\[ f'(x) = e^{x \ln(x)} \cdot \left( 1 \cdot \ln(x) + x \cdot \frac{1}{x} \right) \]

\[ f'(x) = x^x \left( \ln(x) + 1 \right) \]

**Method 2** (Use logarithmic differentiation)

\[ y = x^x \]

\[ \ln(y) = \ln(x^x) \]

\[ \ln(y) = x \ln(x) \]

Original equation

Take logs of both sides

\[ \ln(a^b) = b \ln(a) \]
Now use implicit differentiation.
\[
\frac{1}{y} \cdot y' = \ln(x) + 1
\]

Now solve for \( y' \) and write \( y' \) in terms of \( x \).
\[
y' = y \cdot (\ln(x) + 1)
\]
\[
y' = x^x (\ln(x) + 1)
\]

---

**Ex. 5**

Suppose \( x^3 + y^3 = R^3 \), where \( R \) is a constant. Find \( \frac{d^2y}{dx^2} \).

**Solution:**

Use implicit differentiation to get \( \frac{dy}{dx} \).

\[
x^3 + y^3 = R^3
\]
\[
3x^2 + 3y^2y' = 0
\]
Now solve for $y'$ and use implicit differentiation again.

\[
y' = -\frac{x^2}{y^2}
\]  \hspace{1cm} (1)

Quotient Rule: $F'G - FG'$

\[
y'' = -\left(\frac{2x \cdot y^2 - x^2 \cdot 2y \cdot y'}{y^4}\right)
\]

The answer can have only $x$ and $y$, not $y'$. So just substitute equation (1).

\[
y'' = -\left(\frac{2xy^2 - 2x^2y \left(-\frac{x^2}{y^2}\right)}{y^4}\right)
\]
This simplifies (a lot).

\[ y'' = - \left( \frac{2x y^2 - 2x^2 y}{y^4} \frac{y^2}{y^2} \right) \]

\[ y'' = - \left( \frac{2x y^4 + 2x^4 y}{y^6} \right) \]

\[ y'' = - \left( \frac{2x y (y^3 + x^3)}{y^6} \right) \]

\[ y'' = - \frac{2 R^3 x}{y^5} \]

**Ex. 6**

Find \( f'(x) \).

\[ f(x) = (1 + \sin(2x))^2 \]

**Solution:**
Use logarithmic differentiation

\[ y = (1 + \sin(2x))^x^2 \]

\[ \ln(y) = \ln[(1 + \sin(2x))^x^2] \]

\[ \ln(y) = x^2 \cdot \ln(1 + \sin(2x)) \]

Now use implicit differentiation.

\[ \frac{1}{y} \cdot y' = x^2 \cdot \frac{2 \cos(2x)}{1 + \sin(2x)} + 2x \cdot \ln(1 + \sin(2x)) \]

Now solve for \( y' \) and write \( y \) in terms of \( x \).

\[ y' = y \left( \frac{2x^2 \cos(2x)}{1 + \sin(2x)} + 2x \cdot \ln(1 + \sin(2x)) \right) \]

\[ y' = (1 + \sin(2x))^{x^2} \cdot \left( \frac{2x^2 \cos(2x)}{1 + \sin(2x)} + 2x \cdot \ln(1 + \sin(2x)) \right) \]
**Ex. 7**

Suppose \( x \) and \( y \) are related by the equation:

\[
\sin(x+y) = x + \cos(y)
\]

Find \( \frac{dy}{dx} \).

**Solution:**

Use implicit differentiation.

\[
\sin(x+y) = x + \cos(y)
\]

\[
\cos(x+y) \cdot [1+y'] = 1 - \sin(y) \cdot y'
\]

Calculus is done! Now solve for \( y' \):

\[
\cos(x+y) + \cos(x+y) \cdot y' = 1 - \sin(y) \cdot y'
\]

\[
\sin(y) \cdot y' + \cos(x+y) \cdot y' = 1 - \cos(x+y)
\]

\[
y' \left[ \sin(y) + \cos(x+y) \right] = 1 - \cos(x+y)
\]
Ex. 8

Find \( \frac{dy}{dx} \) if \( \ln (1+xy) = x^2 + 1 \).

Solution:

Use implicit differentiation.

\[
\ln (1+xy) = x^2 + 1
\]

\[
\frac{1}{1+xy} \cdot (xy'+y \cdot 1) = 2x
\]

Calculus is done! Just algebra now!

\[
xy' + y = 2x (1+xy)
\]

\[
xy' = 2x (1+xy) - y
\]

\[
y' = \frac{2x (1+xy) - y}{x}
\]
Review for Exam #1:

Ex. 3

Find an equation of the line tangent to the curve

\[ x^3 + e^{xy} = 3y + 9 \]

at the point \((2, 0)\).

**Solution:**

Use implicit differentiation to find \(y'\):

\[
x^3 + e^{xy} = 3y + 9
\]

\[
3x^2 + e^{xy} \cdot (xy' + 1 \cdot y) = 3y'
\]

Substitute \(x = 2\) and \(y = 0\), then solve for \(y'\):

\[
12 + e^0 (2y' + 1 \cdot 0) = 3y'
\]

\[
12 + 2y' = 3y'
\]

\[
y' = 12 \quad @ \quad (2, 0) \text{ only}
\]

Equation of tangent line:

\[
y - 0 = 12 (x - 2)
\]

Point: \((2, 0)\), Slope: 12
Bonus: What is the normal line?

\[ y - 0 = -\frac{1}{12} (x - 2) \]

point: \((2, 0)\), slope: \(-\frac{1}{12}\)

---

Ex. 2

(a) Given the function \(g(x)\), state the definition of \(g'(4)\).

(b) Let \(F(x) = \frac{1}{3x - 5}\). Find \(F'(2)\) using the limit definition of derivative.

**Solution:**

(a) **Note:** 
- "Slope of tangent line at \(x=4\)"
- "Rate of change of \(g\) at \(x=4\)"
- "Slope of function at \(x=4\)"

ALL WRONG! These are interpretations!
\[ g'(4) = \lim_{h \to 0} \frac{g(4+h) - g(4)}{h} \quad \text{OR} \]
\[ g'(4) = \lim_{x \to 4} \frac{g(x) - g(4)}{x - 4} \]

(b) By definition,
\[ F'(2) = \lim_{h \to 0} \frac{F(2+h) - F(2)}{h} \]
\[ F(x) = \frac{1}{3x - 5} \]

\[
\begin{aligned}
= \lim_{h \to 0} \left( \frac{1}{3(2+h) - 5} - 1 \right) \\
= \lim_{h \to 0} \left( \frac{1}{3h+1} - 1 \right) \cdot \frac{3h+1}{3h+1} \\
= \lim_{h \to 0} \left( \frac{1 - (3h+1)}{h(3h+1)} \right) \\
= \lim_{h \to 0} \left( \frac{-3h}{h(3h+1)} \right) = \lim_{h \to 0} \left( \frac{-3}{3h+1} \right)
\end{aligned}
\]
\[
\lim_{h \to 0} \left( \frac{-3}{3h+1} \right) = \frac{-3}{0+1} = -3
\]

\textbf{Check:}

\[
F(x) = \frac{1}{3x - 5} = (3x - 5)^{-1}
\]

\[
F'(x) = -1 \cdot (3x - 5)^{-2} \cdot 3
\]

\[
F'(2) = -1 \cdot (6 - 5)^{-2} \cdot 3 = -3
\]

\textbf{Ex. 3}

(a) \[
\lim_{x \to 0} \left( \frac{(2x + 9)^2 - 81}{x} \right)
\]

(b) \[
\lim_{x \to 3^-} \left( \frac{|x - 3|}{x - 3} \right)
\]

(c) \[
\lim_{x \to 1} \left( \frac{5 - \sqrt{32 - 7x}}{x - 1} \right)
\]
(a) \[ \lim_{x \to 0} \left( \frac{(2x+9)^2 - 81}{x} \right) \] D.S. of \( x = 0 \) gives \( \frac{0}{0} \)

\[ = \lim_{x \to 0} \left( \frac{4x^2 + 81 + 2 \cdot 2x \cdot 9 - 81}{x} \right) \]

\[ = \lim_{x \to 0} \left( \frac{4x^2 + 36x}{x} \right) = \lim_{x \to 0} (4x + 36) = 36 \]

(b) \[ \lim_{x \to 3^-} \frac{|x-3|}{x-3} = (c.f. 2.2 \& 3.1 \text{ notes}) \]

Q: Is inside of \( |x-3| \) negative or positive?

A: If \( x \to 3^- \), this means \( x \) is close to 3 and \( x < 3 \), or \( x-3 < 0 \). So \( |x-3| = -(x-3) \).

\[ = \lim_{x \to 3^-} \frac{-(x-3)}{x-3} = \lim_{x \to 3^-} (-1) = -1 \]

(c) \[ \lim_{x \to 1} \left( \frac{5 - \sqrt{32-7x}}{x-1} \right) \] D.S. of \( x = 1 \) gives \( \frac{0}{0} \)
\[
\lim_{x \to 1} \left( \frac{5 - \sqrt{32-7x}}{x-1} \cdot \frac{5 + \sqrt{32-7x}}{5 + \sqrt{32-7x}} \right)
\]
\[
= \lim_{x \to 1} \left( \frac{(5)^2 - (\sqrt{32-7x})^2}{(x-1)(5 + \sqrt{32-7x})} \right)
\]
\[
= \lim_{x \to 1} \left( \frac{25 - (32-7x)}{(x-1)(5 + \sqrt{32-7x})} \right) = \frac{25 - 32 + 7x}{(x-1)(5 + \sqrt{32-7x})} = \frac{7x - 7}{(x-1)} = 7(x-1)
\]
\[
= \lim_{x \to 1} \left( \frac{7(x-1)}{(x-1)(5 + \sqrt{32-7x})} \right)
\]
\[
= \lim_{x \to 1} \left( \frac{7}{5 + \sqrt{32-7x}} \right) = \frac{7}{5 + \sqrt{25}} = \frac{7}{10}
\]

**Ex. 4**

Show that the equation

\[x^{2/3} = 2x^2 + 2x - 2\]

has at least one solution in the interval \([0, 1]\). Explain your answer.

**Solution:**

Use Intermediate Value Theorem (IVT). Equivalently, we show the following
equation has a solution in \([0, 1]\).

\[2x^2 + 2x - 2 - x^{2/3} = 0\]

Put \(f(x) = 2x^2 + 2x - 2 - x^{2/3}\). We need to show \(f(x) = 0\) for some \(x\) in \([0, 1]\). Observe:

- \(f(0) = 0 + 0 - 2 - 0 = -2\)
- \(f(1) = 2 + 2 - 2 - 1 = 1\)
- \(f\) is continuous on \([0, 1]\).

(Why? The function \(f\) is a sum of power functions and domain of \(f\) is \((-\infty, \infty)\). So \(f\) is continuous everywhere.)

Since 0 is between -2 and 1, the equation “\(f(x) = 0\)” is satisfied for at least one \(x\)-value in \([0, 1]\) (by the IVT).
Calculate \( \lim_{x \to 1} g(x) \).

**Solution:**

Since \( x = 1 \) is the transition point for \( g(x) \), we must examine the left and right limits separately.

\[
\begin{align*}
g(s) &= \begin{cases} 
3, & s = 1 \\
\sqrt{1-s}, & s < 1 \\
\frac{s^2-s}{s-1}, & s > 1
\end{cases} \\
&= \begin{cases} 
3, & s = 1 \\
\sqrt{1-s}, & s < 1 \\
\frac{s(s-1)}{s-1}, & s > 1
\end{cases} \\
&= \begin{cases} 
3, & s = 1 \\
\sqrt{1-s}, & s < 1 \\
\frac{s(s-1)}{(s-1)}, & s > 1
\end{cases}
\end{align*}
\]

- \( \lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (\sqrt{1-x}) = 0 \)
- \( \lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} \left( \frac{x^2-x}{x-1} \right) = \lim_{x \to 1^+} \frac{x(x-1)}{(x-1)} \)

\[
= \lim_{x \to 1^+} (x) = 1
\]

So \( \lim_{x \to 1} g(x) \) does not exist.
What is the role of \( g(1) = 3 \)?

No role! But it would be relevant for a question on continuity.

**Ex. 6**

\[
f(x) = \begin{cases} 
\frac{\sin(ax)}{x}, & x < 0 \\
2x + 3, & 0 \leq x < 1 \\
b, & x = 1 \\
\frac{x^2 - 1}{x - 1}, & 1 < x
\end{cases}
\]

(a) Find the value of \( a \) so \( f \) is continuous at \( x = 0 \).

(b) Find the value of \( b \) so \( f \) is continuous at \( x = 1 \).

**Solution:**

(a) If \( f \) is to be continuous at \( x = 0 \), the left limit, right limit, and function value must be equal.
\[ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left( \frac{\sin(ax)}{x} \right) = \lim_{x \to 0^-} \left( \frac{\sin(ax)}{a \times x} \times a \right) \]
\[ = \lim_{x \to 0^-} \left( \frac{\sin(ax)}{ax} \right) \times \lim_{x \to 0^-} (a) = a \]
\[ = \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \]

- \[ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2x + 3) = 3 \]

So we must choose \( a = 3 \).

(b) If \( f \) is to be continuous at \( x = 1 \), the left limit, right limit, and function value must be equal.

- \[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x + 3) = 5 \]

- \[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left( \frac{x^2 - 1}{x - 1} \right) = \lim_{x \to 1^+} \frac{(x-1)(x+1)}{x-1} \]
\[ \lim_{x \to 1} (x+1) = 2 \]

\[ f(1) = b \quad \text{(not necessary \ 5 \neq 2)} \]

Since \ 5 \neq 2, no such value of \ b \ exists.

---

**Ex. 7**

- (a) Where is \( f \) not continuous in \((-5,5)\)?
- (b) Where is \( f \) not differentiable in \((-5,5)\)?
- (c) Where is \( f'(x) = 0 \) in \((-5,5)\)?
- (d) Where is \( f'(x) < 0 \) in \((-5,5)\)?

**Solution:**
(a) $x = -3, \ x = -1$
(b) $x = 3, \ x = -3, \ x = -1$

sharp corner \hspace{1cm} f \text{ is discontinuous there}

(c) $x = 1$ and all $x$ in $(-3, -1)$
(d) $(-5, -3) \cup (-1, 1) \cup (3, 5)$

Ex. 8

Find tangent line to $y = 2x^2 - 3x + 1$ at $x = 1$.

Solution:

$f'(x) = 4x - 3$

$f'(1) = 4 - 3 = 1$ (slope)

$f(1) = 2 - 3 + 1 = 0$ (point: $(1, 0)$)

Equation of tangent line:

$y - 0 = 1(x - 1)$

Ex. 9
Find all solutions to the equation:

\[ 2 \ln(x) = \ln \left( \frac{x^5}{5-x} \right) - \ln \left( \frac{x^3}{2+x} \right) \]

**Solution:**

Write each side as a single logarithm.

**LHS:** \( 2 \ln(x) = \ln \left( x^2 \right) \) for \( x > 0 \)

**RHS:**

\[
\ln \left( \frac{x^5}{5-x} \right) = \ln \left( \frac{x^3}{2+x} \right)
\]

So our equation is then

\[ \ln \left( x^2 \right) = \ln \left( \frac{x^2 (2+x)}{5-x} \right) \]
\[
\ln(x^2) = \ln\left(\frac{x^2(2+x)}{5-x}\right)
\]

\[x^2 = \frac{x^2(2+x)}{5-x}\]

\[x^2(5-x) = x^2(2+x)\]

\[x^2(5-x) - x^2(2+x) = 0\]

\[x^2[(5-x) - (2+x)] = 0\]

\[x^2[5-x-2-x] = 0\]

\[x^2(3-2x) = 0\]

\[x^2 = 0 \quad \text{OR} \quad 3-2x = 0\]

\[x = 0 \quad \text{OR} \quad x = \frac{3}{2}\]

Do these solve the original equation?

\[x = 0: \text{ No, since } \ln(0) \text{ is undefined}\]

\[x = \frac{3}{2}: \text{ yes, a solution!}\]
Section 3.7: Related Rates

Note: All variables are assumed to be function of time $t$.

Ex. 1

A ladder of length $L = 10\text{ft}$ is leaning against a wall. Suppose the bottom of the ladder slides away from the wall at $2\text{ft/s}$. How fast and in what direction is the top of the ladder moving when the top is $8\text{ft}$ from the ground?

Solution:

We need an equation that relates $x$ and $y$ and is true for all time $t$. 

$x$ We assume ladder never loses contact with wall or ground
\[ (1) \quad x^2 + y^2 = 100 \]

Now we parse each English sentence into a mathematical sentence (equations) that tells us information that is true only at one specific time.

**Given**

"\( x \) is changing at a rate of 2 ft/s"

\[ \frac{dx}{dt} = 2 \]

**Want**

\[ \frac{dy}{dt} = ??? \]

@ time when \( y = 8 \)

How do we introduce \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \) into our model? Differentiate (1) with respect to time \( t \).

(1) \[ x^2 + y^2 = 100 \]

(2) \[ 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0 \]

→ Implicit differentiation wrt. time \( t \).

**Reminder**
\[
\frac{\text{d}}{\text{d}t} (x^2) = \frac{\text{d}}{\text{d}t} \left( [x(t)]^2 \right) = 2(x(t)) \frac{\text{d}x}{\text{d}t}.
\]

Equations (1) and (2) are true for all time \( t \). Now we substitute our given information specific to one time only (snapshot time).

Put \( \frac{\text{d}x}{\text{d}t} = 2 \) and \( y = 8 \) in (1) and (2).

(1)* \[x^2 + 64 = 100\]

(2)* \[4x + 16 \frac{\text{d}y}{\text{d}t} = 0\]

These equations are true only at the snapshot time.

Now solve for \( \frac{\text{d}y}{\text{d}t} \). Equation (1)* gives \( x = 6 \). Putting \( x = 6 \) into (2)*:

\[24 + 16 \frac{\text{d}y}{\text{d}t} = 0 \quad \Rightarrow \quad \frac{\text{d}y}{\text{d}t} = -\frac{3}{2}\]

The top of the ladder is moving (when \( y = 8 \)) at \( \frac{3}{2} \) ft/s down the
wall. (The minus sign on $-\frac{3}{2}$ tells you $y$ is decreasing.)

**General Strategy for Related Rates**

1. Draw diagram and label variables. (Identify constants and variables.)
2. Write down all equations relating the variables, which are true for all time.
3. Differentiate all equations from step 2 implicitly with respect to time.
4. Substitute any given values, which hold only at one specific time.
5. Solve for desired values (correct units!)

**Ex. 2** (Spring 2018 Exam #2)

Total surface area of a cube is changing at a rate of $12 \text{ in}^2/\text{s}$, when the length
of one of its sides is 10 in. At what rate is the volume of the cube changing at that time?

Solution:

Variables: \( x, A, V \)

\[ V = x^3 \] (1)

\[ A = 6x^2 \] (2)

\( \Rightarrow \) Note: Do not label \( x = 10 \)!

(\( x \) changes over time)

Now parse each English sentence into a mathematical sentence (equation)

Given

Want @ snapshot

\[ \frac{dA}{dt} = 12 \]

\[ \frac{dV}{dt} = ??? \]

@ time when \( x = 10 \)

Now differentiate (1) and (2) wrt. \( t \).

(1) \[ V = x^3 \] \( \frac{dv}{dt} = 3x^2 \cdot \frac{dx}{dt} \] (3)
These four equations are true for all times.

Now substitute information specific to the snapshot time. Put \( \frac{dA}{dt} = 12 \) and \( x = 10 \) in to Equations (1) – (4).

(1)* \( V = 1000 \quad \frac{dV}{dt} = 300 \cdot \frac{dx}{dt} \) (3)*

(2)* \( A = 600 \quad 12 = 120 \cdot \frac{dx}{dt} \) (4)*

These four equations are true only at the snapshot time.

Now solve for \( \frac{dV}{dt} \). From (4)* we get \( \frac{dx}{dt} = \frac{1}{10} \). Putting \( \frac{dx}{dt} = \frac{1}{10} \) into (3)* gives \( \frac{dV}{dt} = 300 \cdot \frac{1}{10} = 30 \).

The volume is changing at a rate
A 5ft-tall person stands still 8 feet from point P, which is directly below a lantern. At the moment when the lantern is 15 feet above the ground, the lantern is falling at a rate of 4 ft/s. At what rate is the length of the person's shadow changing at this moment?

Solution:

We need an equation that relates $x$. We must not label this as "15" since $h$ changes over time.

$x = \text{length of shadow}$
and \( h \). We use similar triangles.

\[
\frac{\text{large height}}{\text{large base}} = \frac{\text{small height}}{\text{small base}}
\]

\[
\frac{h}{x + 8} = \frac{5}{x}
\]

Rearrange this equation slightly:

\[
h = \frac{5(x + 8)}{x} = \frac{5x + 40}{x} = 5 + \frac{40}{x}
\]  \( (1) \)

Now we parse each English sentence into an equation.

<table>
<thead>
<tr>
<th>Given</th>
<th>Want</th>
<th>@ snapshot</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dh}{dt} = -4 )</td>
<td>( \frac{dx}{dt} = ??? )</td>
<td></td>
</tr>
<tr>
<td>@ time when ( h = 15 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Negative value since \( h \) is decreasing (lantern is falling)

Now differentiate \((1)\) wrt. time \( t \).
\[
(1) \quad h = 5 + \frac{40}{x} \quad \left( h = 5 + 40x^{-1} \right)
\]
\[
(2) \quad \frac{dh}{dt} = -\frac{40}{x^2} \cdot \frac{dx}{dt} \quad \left( \frac{dh}{dt} = 0 - 40x^{-2} \cdot \frac{dx}{dt} \right)
\]

Now substitute information specific to snapshot time. Put \( \frac{dh}{dt} = -4 \) and \( h = 15 \) into (1) and (2).

\[
(1)^* \quad 15 = 5 + \frac{40}{x}
\]

\[
(2)^* \quad -4 = -\frac{40}{x^2} \cdot \frac{dx}{dt}
\]

Now solve for \( \frac{dx}{dt} \). From (1)*, we get \( x = 4 \). Putting \( x = 4 \) into (2)* gives

\[
-4 = -\frac{40}{16} \cdot \frac{dx}{dt} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{8}{5} \text{ ft/s}
\]

Length of shadow increases at \( \frac{8}{5} \text{ ft/s} \).
Section 4.2: Mean Value Theorem

What types of functions are continuous but not differentiable? (c.f., 3.1)

(In 3.1, we covered how to recognize such functions graphically. The following notes describe how to recognize such functions analytically. These notes were summarized at the end of notes for 3.1.)

There are three categories to look out for:

0. **Absolute Value**: \( f(x) = |x| \)

The function \( f \) is not differentiable at \( x = 0 \). The graph of \( f \) has a sharp corner at \( x = 0 \). So the function \( h(x) = |g(x)| \) is (possibly) not differentiable when \( g(x) = 0 \).
Ex: The function $h(x) = |x^2 - 4|$ is not differentiable at $x = -2$ and $x = 2$ ($x^2 - 4 = 0 \Rightarrow x = \pm 2$)

2) Power Functions: $f(x) = x^n$ (Exponents less than 1) ($0 < n < 1$)

The function $f$ is not differentiable at $x = 0$. The graph of $f$ has a cusp or vertical tangent line at $x = 0$.

So the function $h(x) = g(x)^n$ is (possibly) not differentiable when $g(x) = 0$.

Ex: The function $h(x) = (x^2 - 4)^{2/3}$ is not differentiable at $x = -2$ and $x = 2$ ($x^2 - 4 = 0 \Rightarrow x = \pm 2$)
Similarly, \( h(x) = (x^2 - 4)^{\frac{1}{3}} \) is not differentiable at \( x = -2 \) and \( x = 2 \); \( h(x) = (x^2 - 4)^{\frac{1}{2}} \) is also not differentiable at \( x = -2 \) and \( x = 2 \).

\[
y = (x^2 - 4)^{\frac{2}{3}} \quad y = (x^2 - 4)^{\frac{1}{3}} \quad y = (x^2 - 4)^{\frac{1}{2}}
\]

**3. Piecewise-Defined Functions**

(assume each piece is continuous)

Very often (but not always), a piecewise-defined function is not differentiable at the transition points.

\[
y = \begin{cases} 
-2x & \text{if } x < 0 \\
\frac{x^2}{4} & \text{if } x \geq 0
\end{cases}
\]

Not diff. at \( x = 0 \)

\[
y = \begin{cases} 
1 - \cos(x) & \text{if } x < 0 \\
x^2 & \text{if } x \geq 0
\end{cases}
\]

Differentiable at \( x = 0 \)
**Thm:** (Mean Value Theorem, MVT)

Suppose \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\).

Then there exists \( c \) in \((a, b)\) with

\[
    f'(c) = \frac{f(b) - f(a)}{b - a}
\]

**Graphical Interpretation of MVT**

**Ex. 1**

Let \( f(x) = x^{2/3} \). For each interval, determine whether the hypotheses of the MVT are satisfied. If yes, find
all values of $c$ guaranteed to exist by the MVT.

(a) $[-1, 1]$

(b) $[0, 8]$

Solution:

- Where is $f$ continuous?
  
  Power functions are cont. on their domain, so $f$ is cont. on $(-\infty, \infty)$. So the continuity hypothesis in MVT is always satisfied for $f$.

- Where is $f$ differentiable?
  
  Everywhere except $x = 0$. ($0 < \frac{2}{3} < 1$)
  So any open interval containing $x = 0$ does not satisfy the MVT hypothesis.

(a) Since $0$ is in $(-1, 1)$, hypotheses of MVT are not satisfied.

(b) Since $0$ is not in $(0, 8)$, hypotheses of MVT are satisfied. So there exists
c in \((0, 8)\) such that

\[ f'(c) = \frac{f(8) - f(0)}{8 - 0} \quad \left[ a = 0 \quad b = 8 \right] \]

Now calculate \(f'(c)\) and solve this equation for \(c\).

\[ \frac{2}{3} c^{-1/3} = \frac{8^{2/3} - 0^{2/3}}{8 - 0} \]

\[ \frac{2}{3} c^{-1/3} = 8^{-1/3} \]

\[ c = \left(\frac{3}{2}\right)^{-3}. 8 = \frac{64}{27} \]

---

**Important Special Case of MVT**

**Thm:** (Rolle's Theorem)

Suppose \(f\) is continuous on \([a, b]\), differentiable on \((a, b)\), and \(f(a) = f(b)\). Then there exists \(c\) in \((a, b)\) with

\[ f'(c) = 0 \quad \left(= \frac{f(b) - f(a)}{b - a}\right) \]
Section 4.1: Extreme Values

Some definitions

**ABSOLUTE vs. RELATIVE**

<table>
<thead>
<tr>
<th>Absolute minimum value of ( f ) on ([a, b])</th>
<th>Relative minimum value of ( f ) at ( x = c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( f(c) ) is the abs. min. value of ( f ) on ([a, b]), then ( f(c) ) is the smallest possible value of ( f ) for all ( x ) in ([a, b])</td>
<td>If ( f(c) ) is a rel. min. value of ( f ), then ( f(c) ) is the smallest possible value of ( f ) for ( x ) near ( c ).</td>
</tr>
</tbody>
</table>

We have similar definitions for absolute maximum and relative maximum (Replace “smallest” with “largest”)

Synonyms: Absolute min. \( \Leftrightarrow \) Global min.

Relative min. \( \Leftrightarrow \) Local min.

“Extremum”: maximum or minimum

Locating Extreme Values Graphically
Relative Minimum Values

\[ f(C), \ f(E), \ f(D) \]

Also the abs. minimum value

Relative Maximum Values

\[ f(B), \ f(D), \ f(F) \]

Your textbook does not allow local extrema at an end point

However, \( f(F) \) is the abs. max value, and your textbook does allow absolute
extrema to occur endpoints.

**Thm:** (Extreme Value Theorem, EVT) Suppose \( f \) is continuous on the closed, bounded interval \([a, b]\). Then the absolute min. and max. of \( f \) on \([a, b]\) exist.

What can go wrong if \( f \) is not continuous?

```
      |      | Range of \( f \):
      |      | (1, 5)
      |      |
      |      |
  5   |      |
      |      |
      |      |
  1   |      |
      |      | Note: \( f \) is discontinuous on \([a, b]\).

The absolute min. value does not exist.
The absolute max. value does not exist.
```

**Def:** A number \( c \) in the domain of \( f \) is a critical number if either \( f'(c) \) does not exist or \( f'(c) = 0 \).
Thm: (Fermat)
If \( f(c) \) is a local extremum, then \( c \) is a critical number of \( f \).

So these theorems tell us that if we want to find local extrema, we should only consider critical numbers or endpoints.

Finding absolute min/max of \( f \) on \([a, b]\).

- Is \( f \) continuous on \([a, b]\)?
  - \( \iff \) no, stop. (EVT concludes nothing.)
  - \( \iff \) yes, then ......

- Find critical numbers of \( f \) on \((a, b)\).
  - for which \( x \)-values does \( f'(x) \) dne?
  - solve the equation \( f'(x) = 0 \)

- Construct a list of candidate extreme values (values of \( f \) at critical \#’s and values of \( f \) and endpoints)

- Smallest value is absolute min.
  - Largest value is absolute max.
Ex 1

Find the absolute extreme values of
\[ f(x) = x^3 - 6x^2 + 8 \]
on \([1, 6]\).

Solution:
(Note that \( f \) is continuous on \([1, 6]\). So EVT guarantees that the absolute min. and max. exist.)

First find the critical numbers of \( f \).

- \( f'(x) \) dne: none
- \( f'(x) = 0 \):
  \[ 3x^2 - 12x = 0 \]
  \[ 3x(x - 4) = 0 \]
  \[ x = 0 \quad \text{or} \quad x = 4 \]
  \( \not{x = 0} \) or \( x = 4 \) not in \([1, 6]\)

Now construct list of candidate values
\[ y = x^3 - 6x^2 + 8 = x^2(x - 6) + 8 \]
\begin{align*}
\text{x-value} & \quad \text{y-value} \\
\end{align*}
critical # 4 \quad 16(-2) + 8 = -24

endpoint 1 \quad 1(-5) + 8 = 3

endpoint 6 \quad 36(0) + 8 = 8

The absolute min. of \( f \) on \([1,6]\) is -24.
The absolute max. of \( f \) on \([1,6]\) is 8

---

Ex. 2

Find the absolute extreme values of

\[ f(x) = (x^2 - 16)^{2/3} + 20 \]
on \([-5,5]\).

Solution:

Find the critical #’s of \( f \):

- \( f'(x) \) dne: Exponent less than 1!
  (Recall notes from 4.2: if \( 0 < N < 1 \), then \( g(x)^N \) is not differentiable where \( g(x) = 0 \)
  \[ x^2 - 16 = 0 \]
  \[ x = -4 \quad \text{or} \quad x = 4 \]

- \( f'(x) = 0 \):
\[
\frac{2}{3} (x^2 - 16)^{-\frac{1}{3}} \cdot (2x) = 0
\]
\[
2x = 0
\]
\[
x = 0
\]

Now construct a list of candidate values

\[
y = (x^2 - 16)^{\frac{2}{3}} + 20
\]

<table>
<thead>
<tr>
<th>x-values</th>
<th>y-values</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>(0^{2/3} + 20) = 20</td>
</tr>
<tr>
<td>4</td>
<td>(0^{2/3} + 20) = 20</td>
</tr>
<tr>
<td>0</td>
<td>((-16)^{2/3} + 20) = (16^{2/3} + 20)</td>
</tr>
<tr>
<td>-5</td>
<td>= (9^{2/3} + 20)</td>
</tr>
<tr>
<td>5</td>
<td>= (9^{2/3} + 20)</td>
</tr>
</tbody>
</table>

The absolute min. of \(f\) on \([-5, 5]\) is 20. The absolute max. of \(f\) on \([-5, 5]\) is \(16^{2/3} + 20\). (It is okay that the absolute minimum occurs at multiple x-values.)
Find absolute extreme values of 
\[ f(x) = x - \frac{4x}{x+1} \]
on \([0, 3]\).

**Solution:**
(Note: \( f \) is continuous on \([0, 3]\).)

**Critical numbers of \( f \):**
- \( f'(x) \) does: none
  \((x=-1 \text{ is not in } [0, 3], \text{ but } x=-1 \text{ is not a critical # anyway since } -1 \text{ is not in the domain of } f.)\)
- \( f'(x) = 0 \):
  \[
  1 - \frac{4}{(x+1)^2} = 0
  \]
  \[
  (x+1)^2 = 4
  \]
  \[
  x+1 = -2 \quad \text{or} \quad x+1 = 2
  \]
  \[
  x = -3 \quad \text{or} \quad x = 1
  \]

not in \([0, 3]\)

Now make candidate list of values.
\[ y = x - \frac{4x}{x+1} \]

<table>
<thead>
<tr>
<th>x-values</th>
<th>y-values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Critical #</td>
<td>1</td>
</tr>
<tr>
<td>endpoints</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
</tbody>
</table>

The absolute min. of \(f\) on \([0, 3]\) is \(-1\).

The absolute max. of \(f\) on \([0, 3]\) is \(0\).
Section 4.3: Shapes of Graphs

What does $f'$ say about the graph of $f(x)$?

- If $f'(x) > 0$, $f$ is increasing (as $x$ increases, $y$ increases).
- If $f'(x) < 0$, $f$ is decreasing (as $x$ increases, $y$ decreases).

What about local extreme values?

Suppose $c$ is a critical # of $f$.

- If $f'(x) > 0$, $f$ has no local extreme value. $f'(x)$ does not change sign at $x = c$. 
- If $f'(c) = 0$, $f$ may have a local extreme value at $x = c$. 
- If $f'(c)$ changes sign, $f$ has a local extreme value at $x = c$. 

**Summary of Info from \( f'(x) \):**

- **Intervals of Increase / Decrease**

<table>
<thead>
<tr>
<th>Sign of ( f'(x) ) for all ( x ) in ((a,b))</th>
<th>Shape of graph of ( y = f(x) ) on ((a,b))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ominus )</td>
<td>decreasing</td>
</tr>
<tr>
<td>( \oplus )</td>
<td>increasing</td>
</tr>
</tbody>
</table>

- **Local Extreme Values**
  (First Derivative Test)

Suppose \( c \) is a critical # of \( f \).
(\( \text{So either } f'(c) = 0 \text{ or } f'(c) \text{ dne.} \))
<table>
<thead>
<tr>
<th>Change in sign of $f'(x)$ at $x=c$</th>
<th>Classification of the value $f(c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>- to +</td>
<td>local minimum</td>
</tr>
<tr>
<td>+ to -</td>
<td>local maximum</td>
</tr>
<tr>
<td>no change</td>
<td>not a local extremum</td>
</tr>
</tbody>
</table>

What does $f''$ say about the graph of $f(x)$?

Note: $f'(x) > 0$ for both graphs below. But how is $f'(x)$ changing?

- $f'(x)$ is decreasing $f''(x) < 0$
  - concave down
  - (graph of $f$ lies below tangent lines)

- $f'(x)$ is increasing $f''(x) > 0$
  - concave up
  - (graph of $f$ lies above tangent lines)

Can $f''$ tell us about local extreme values?
(Yes, as long as $f''$ is continuous)

Suppose $f'(c) = 0$ and $f''$ is continuous.

![Graph showing local minimum and local maximum points.]

**Summary of Info from $f''(x)$:**

- **Intervals of Concavity**

<table>
<thead>
<tr>
<th>Sign of $f''(x)$ for all $x$ in $(a, b)$</th>
<th>Shape of graph of $y = f(x)$ on $(a, b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt;$</td>
<td>concave down</td>
</tr>
<tr>
<td>$&gt;$</td>
<td>concave up</td>
</tr>
</tbody>
</table>

- **Inflection Points:**

If concavity of $f$ changes at $x = c$ and $f$ is continuous at $x = c$, then $(c, f(c))$ is an inflection point.
Inflection points can occur only where either \( f''(x) \) dne. or \( f''(x) = 0 \).

- **Local Extreme Values**
  - (Second Derivative Test)
  - Suppose \( f'(c) = 0 \) and \( f''(c) \) is cont. at \( x = c \)

<table>
<thead>
<tr>
<th>Sign of ( f''(c) ) (must have ( f'(c) = 0 ))</th>
<th>Classification of the value ( f(c) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( + )</td>
<td>local minimum</td>
</tr>
<tr>
<td>( - )</td>
<td>local maximum</td>
</tr>
<tr>
<td>zero</td>
<td>test inconclusive</td>
</tr>
</tbody>
</table>

**How to graph** \( y = f(x) \)
• Information from $f(x)$:
  • points on graph
  • vertical asymptotes
  • horizontal asymptotes
  \[ \text{Section 4.4} \]

• Information from $f'(x)$:
  Find first-order critical #\(s\):
  • $f'(x)$ \(\text{dne}\).
  • $f'(x) = 0$
  Construct a sign chart for $f'(x)$
  (cut points are first-order critical #\(s\) and x-values of vertical asymptotes)
  • intervals of increase/decrease
  • local extreme values

• Information from $f''(x)$:
  Find second-order critical #\(s\):
  • $f''(x)$ \(\text{dne}\).
  • $f''(x) = 0$
  Construct a sign chart for $f''(x)$
(cut points are second-order critical #’s and x-values of vertical asymptotes)

• intervals of concavity
• inflection points
• (optional: local extreme values)

• Graph \( y = f(x) \)

• List important points
  (local min, local max, inflection points)

• Summarize all info from \( f, f', \) and \( f'' \).

• Then use chart below.

```
<table>
<thead>
<tr>
<th>Concave down</th>
<th>Concave up</th>
</tr>
</thead>
<tbody>
<tr>
<td>decreasing</td>
<td>increasing</td>
</tr>
</tbody>
</table>
```

**Ex. 1**

Graph \( f(x) = x^3 - 12x^2 \) on \([-1, 9]\).
**Solution:**

\[ f(x) = x^3 - 12x^2 = x^2(x-12) \]
\[ f'(x) = 3x^2 - 24x = 3x(x-8) \]
\[ f''(x) = 6x - 24 = 6(x-4) \]

1. **Information from \( f(x) \):**
   - (Polynomials have no asymptotes)

2. **Information from \( f'(x) \):**
   - First-order critical #'s:
     - (Recall \( f'(x) = 3x(x-8) \))
     - \( f'(x) \) = none
     - \( f'(x) = 0 \): \( x=0, \ x=8 \)

Now construct sign chart for \( f'(x) \):

\[ f'(x) = 3x(x-8) \]
\[ f'(-1) = -15 \]
\[ f'(1) = + \quad - \quad = = - \]
\[ f'(9) = + \quad + \quad = + \]

\( f \) is decreasing on \([0, 8]\)

acceptable: \((0, 8)\)

\( f \) is increasing on \((-\infty, 0], [8, \infty)\)

\(!\) acceptable: \((-\infty, 0) \cup (8, \infty)\)

local min @ \( x = 8 \)

local max @ \( x = 0 \)

**3. Information from \( f''(x) \):**

Second-order critical points:

(Recall \( f''(x) = 6(x-4) \))

- \( f''(x) \) due: none
- \( f''(x) = 0: x = 4 \)

Now construct sign chart for \( f''(x) \):

\[ f''(x) = 6(x-4) \]

\begin{align*}
\text{Shape of } f & \quad \text{Sign of } f'' \\
\begin{array}{c}
\begin{array}{c}
\text{test point}
\end{array}
\end{array}
\end{align*}
\[-f''(0) = \bigcirc \bigcirc = \bigcirc\]
\[-f''(5) = \bigcirc \bigcirc = \bigcirc\]

\(f\) is concave down on \((-\infty, 4]\)
acceptable: \((-\infty, 4]\)

\(f\) is concave up on \([4, \infty)\)
acceptable: \((4, \infty)\)

Inflection point(s) @ \(x = 4\)

**Graph** \(y = f(x)\):
Recall \(f(x) = x^2(x-12)\) on \([-1, 9]\)

Important Points on Graph:

<table>
<thead>
<tr>
<th>(x)-value</th>
<th>(y)-value</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-13</td>
<td>endpoint</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>local max</td>
</tr>
<tr>
<td>4</td>
<td>-128</td>
<td>inflection point</td>
</tr>
<tr>
<td>8</td>
<td>-256</td>
<td>local min</td>
</tr>
<tr>
<td>9</td>
<td>-243</td>
<td>endpoint</td>
</tr>
</tbody>
</table>

Summary of Info from \(f, f', \text{and } f''\):
Ex. 2

Graph $f(x) = x(x-2)^3$ on $[-1, 3]$.

Solution:

$f(x) = x(x-2)^3$
\( f''(x) = 1 \cdot (x-2)^3 + x \cdot 3(x-2)^2 \cdot 1 \)
\( = ( (x-2) + 3x ) (x-2)^2 \)
\( = 2(2x-1) (x-2)^2 \)

\( f''(x) = 2 \left[ 2 \cdot (x-2)^2 + (2x-1) \cdot 2 (x-2) \cdot 1 \right] \)
\( = 2 (x-2) \left( 2 (x-2) + 2 (2x-1) \right) \)
\( = 12 (x-2) (x-1) \)

(Don't expand. Just use product rule.)

1. Information from \( f(x) \):
   (Polynomials have no asymptotes)

2. Information from \( f'(x) \):
   First-order critical #s:
   (Recall \( f'(x) = 2 (2x-1)(x-2)^2 \))
   - \( f'(x) \) dne: none
   - \( f'(x) = 0 \): \( x = \frac{1}{2}, x = 2 \)

Now construct sign chart for \( f'(x) \):

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( \frac{1}{2} )</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td></td>
</tr>
</tbody>
</table>

Sign of \( f' \)

Shape of \( f \)

Test point
\[ f'(x) = 2(2x-1)(x-2)^2 \]

\[ f'(0) = + \quad + \quad + = + \]
\[ f'(1) = + \quad + \quad + = + \]
\[ f'(3) = + \quad + \quad + = + \]

\( f \) is decreasing on \((-\infty, \frac{1}{2}]\)
\( f \) is increasing on \([\frac{1}{2}, \infty)\)

Not two separate intervals!

Local min @ \( x = \frac{1}{2} \)

Local max @ nowhere

3) Information from \( f''(x) \):

Second-order critical points:
(Recall \( f''(x) = 12(x-2)(x-1) \))

- \( f''(x) \) due: none
- \( f''(x) = 0 \): \( x = 1, x = 2 \)

Now construct sign chart for \( f''(x) \):

\[ f''(x) = 12(x-2)(x-1) \]
\[ f''(0) = - \quad - \quad - \quad + \]  
\[ f''(1) = - \quad + \quad - \quad + \]  
\[ f''(3) = + \quad + \quad + \quad + \]

\( f \) is concave down on \([1, 2]\)

\( f \) is concave up on \((-\infty, 1], [2, \infty)\)

Inflection point(s) @ \( x = 1, x = 2 \)

Graph \( y = f(x) \):
Recall \( f(x) = x(x-2)^3 \) on \([-1, 3]\)

**Important Points on Graph:**

<table>
<thead>
<tr>
<th>x-value</th>
<th>y-value</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>27</td>
<td>endpoint</td>
</tr>
<tr>
<td>1/2</td>
<td>-27/16</td>
<td>local min</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>inflection point</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>inflection point</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>endpoint</td>
</tr>
</tbody>
</table>

Summary of Info from \( f, f', \) and \( f'' \):

-1 \quad \frac{1}{2} \quad 1 \quad 2 \quad 3

\( \text{inc/dec} \quad \text{dec.} \quad \text{inc.} \quad \text{increasing} \)
(Make sure to label important points. Graph does not have to be to scale.)
Section 4.4: Asymptotes

Consider the following limit. What does it mean?

As \( x \) gets arbitrarily large and positive...

\[
\lim_{x \to \infty} \frac{1}{x} = 0 \quad \rightarrow \quad \text{values of } \frac{1}{x} \text{ become arbitrarily close to 0.}
\]

... what happens to the values of \( \frac{1}{x} \)?
Do these values approach some #?

Similar limit as \( x \to -\infty \)....

As \( x \) gets arbitrarily large and negative...

\[
\lim_{x \to -\infty} \frac{1}{x} = 0 \quad \rightarrow \quad \text{values of } \frac{1}{x} \text{ become arbitrarily close to 0.}
\]

... what happens to the values of \( \frac{1}{x} \)?
Do these values approach some #?

Now consider a general case. Suppose \( n > 0 \):

\[
\lim_{x \to \infty} \frac{1}{x^n} = \lim_{x \to \infty} \left( \frac{1}{x} \right)^n = \left( \lim_{x \to \infty} \frac{1}{x} \right)^n = 0^n = 0
\]

\[
\lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0, \quad \lim_{x \to \infty} \frac{5}{x^3} = 0
\]
Master Strategy for Limits with $x \to \pm \infty$

If $x \to \infty$ or if $x \to -\infty$, then factor out "highest power" of numerator and denominator separately.

**Ex. 1**

Calculate $\lim_{x \to \infty} \frac{2x}{x^2 + 1}$.

**Solution:**

$$\lim_{x \to \infty} \frac{2x}{x^2 + 1} = \lim_{x \to \infty} \left(\frac{x}{x^2} \cdot \frac{2}{1 + \frac{1}{x^2}}\right)$$

Partial cancellation

Leftover terms

$$= \lim_{x \to \infty} \left(\frac{\frac{1}{x}}{1 + \frac{1}{x^2}} \cdot \frac{2}{1 + \frac{1}{x^2}}\right) = 0 \cdot \frac{2}{1 + 0} = 0$$

These terms have limit 0 as $x \to \infty$.

**Ex. 2**
Calculate \( \lim_{x \to \infty} \frac{3x^2 - 5x + 1}{4x^2 - 7} \).

**Solution:**
\[
\lim_{x \to \infty} \frac{3x^2 - 5x + 1}{4x^2 - 7} = \lim_{x \to \infty} \left( \frac{x^2}{x^2} \cdot \frac{3 - \frac{5}{x} + \frac{1}{x^2}}{4 - \frac{7}{x^2}} \right) = \lim_{x \to \infty} \left(1 \cdot \frac{3 - 0 + 0}{4 - 0}\right) = \frac{3}{4}
\]

marked terms have limit 0 as \(x \to \infty\)

---

**Ex. 3**

Calculate \( \lim_{x \to -\infty} \frac{5x^3 - 2x}{x^2 + 1} \).

**Solution:**
\[
\lim_{x \to -\infty} \frac{5x^3 - 2x}{x^2 + 1} = \lim_{x \to -\infty} \left( \frac{x^3}{x^2} \cdot \frac{5 - \frac{2}{x^2}}{1 + \frac{1}{x^2}} \right) = \lim_{x \to -\infty} \left( x \cdot \frac{5 - 2/x^2}{1 + 1/x^2} \right) = (-\infty) \cdot \frac{5 - 0}{1 + 0} = -\infty
\]
marked terms have
limit 0 as \( x \to -\infty \)

\[ (\infty) \cdot 5 = -\infty \]

---

**Ex. 4**

Calculate \( \lim_{x \to -\infty} \frac{\sqrt{25x^2 + 3}}{9x - 1} \).

**Solution:**

**Note:** \( \sqrt{a + b} \neq \sqrt{a} + \sqrt{b} \)

\[
\lim_{x \to -\infty} \frac{\sqrt{25x^2 + 3}}{9x - 1} = \lim_{x \to -\infty} \frac{\sqrt{x^2 (25 + \frac{3}{x^2})}}{x (9 - \frac{1}{x})}
\]

**Note:** If \( a, b > 0 \), then \( \sqrt{ab} = \sqrt{a} \sqrt{b} \)

\[
= \lim_{x \to -\infty} \left( \frac{\sqrt{x^2}}{x} \cdot \frac{\sqrt{25 + \frac{3}{x^2}}}{9 - \frac{1}{x}} \right)
\]

**Note:** \( \sqrt{x^2} \neq x \)

\( \sqrt{\text{must return a positive #!}} \quad x = -1? \)

\( \sqrt{x^2} \neq \pm x \)
\[ \sqrt{x^2} = |x| \]

L.1 forces output to be \( \geq 0 \).

\[
\lim_{x \to -\infty} \left( \frac{|x|}{x} \cdot \frac{\sqrt{25 + \frac{3}{x^2}}}{9 - \frac{1}{x}} \right)
\]

Since \( x \to -\infty \), we may assume \( x < 0 \). So \( |x| = -x \).

\[
\lim_{x \to -\infty} \left( \frac{-x}{x} \cdot \frac{\sqrt{25 + \frac{3}{x^2}}}{9 - \frac{1}{x}} \right)
\]

\[
\lim_{x \to -\infty} \left( -1 \cdot \frac{\sqrt{25 + \frac{3}{x^2}}}{9 - \frac{1}{x}} \right) \text{ marked terms have limit } 0 \text{ as } x \to \infty
\]

\[
= -1 \cdot \frac{\sqrt{25} + 0}{9 - 0} = -\frac{5}{9}
\]

**Bonus: (Different Exercise)**

\[
\lim_{x \to \infty} \left( \frac{\sqrt{25x^2 + 3}}{9x - 1} \right) = \lim_{x \to \infty} \left( \frac{\sqrt{25 + \frac{3}{x^2}}}{9 - \frac{1}{x}} \right) = \frac{5}{9}
\]

\[ |x| = x \text{ since } x \to \infty \]
Some Special Limits from Section 2.4

\[ y = e^x \]

\[ \lim_{x \to -\infty} e^x = 0 \]
\[ \lim_{x \to \infty} e^x = \infty \]

\[ y = e^{-x} \]

\[ \lim_{x \to -\infty} e^{-x} = \infty \]
\[ \lim_{x \to \infty} e^{-x} = 0 \]

Ex. 5:

Let \( f(x) = \frac{3 + e^x}{5 - 4e^x} \). Calculate

(a) \( \lim_{x \to -\infty} f(x) \)

(b) \( \lim_{x \to \infty} f(x) \)

Solution:

(a) \( \lim_{x \to -\infty} \frac{3 + e^x}{5 - 4e^x} = \frac{3 + 0}{5 - 0} = \frac{3}{5} \)

Recall: \( \lim_{x \to -\infty} e^x = 0 \)
(b) \( \lim_{x \to \infty} \frac{3 + e^x}{5 - 4e^x} \neq \frac{\frac{3 + \infty}{\infty}}{\frac{5 - 4 \cdot \infty}{-\infty}} \neq -1 \)

Recall: \( \lim_{x \to \infty} e^x = \infty \) "\( \infty \)/" is undefined

Instead, use the master strategy of factoring out "highest power" (largest term as \( x \to \infty \)).

\[
\lim_{x \to \infty} \frac{3 + e^x}{5 - 4e^x} = \lim_{x \to \infty} \left( \frac{e^x}{e^x} \cdot \frac{3e^{-x} + 1}{5e^{-x} - 4} \right)
\]

\[
= \lim_{x \to \infty} \left( 1 \cdot \frac{3e^{-x} + 1}{5e^{-x} - 4} \right) = 1 \cdot \frac{0 + 1}{0 - 4} = -\frac{1}{4}
\]

Recall: \( \lim_{x \to \infty} e^{-x} = 0 \)

Alternatively,

\[
= \lim_{x \to \infty} \left( \frac{\frac{3}{e^x} + 1}{\frac{5}{e^x} - 4} \right) = \frac{\frac{3}{8} + 1}{\frac{5}{8} - 4} = \frac{0 + 1}{0 - 4} = -\frac{1}{4}
\]

Recall: \( \lim_{x \to \infty} e^x = \infty \) "finite #" = 0"
Now we consider limits which are infinite:

\[ \lim_{x \to 0^+} \frac{1}{x} = \infty \]

This means \( x \) is a small positive \( \# \).
So \( \frac{1}{x} \) is a large positive \( \# \).
So we say that \( \frac{1}{x} \to \infty \) as \( x \to 0^+ \).

\[ \lim_{x \to 0^-} \frac{1}{x} = -\infty \]

This means \( x \) is a small negative \( \# \).
So \( \frac{1}{x} \) is a large negative \( \# \).
So we say that \( \frac{1}{x} \to -\infty \) as \( x \to 0^- \).

**Master Strategy for Infinite Limits**

If D.S. gives the expression “**nonzero \( \# \)**”
then each one-sided limit is **infinite**.
To determine whether limit is \( \infty \) or \( -\infty \), we perform a sign analysis of numerator and denominator.

**Horizontal Asymptotes**

(\( L \) is finite)

If either \( \lim_{x \to -\infty} f(x) = L \) or \( \lim_{x \to \infty} f(x) = L \), then the line \( y = L \) is a horizontal asymptote (HA) for \( y = f(x) \).

- 0 HA's
- 1 HA
- 2 HA's
- 1 HA
Vertical Asymptotes

If either \( \lim_{x \to a^-} f(x) \) or \( \lim_{x \to a^+} f(x) \) is infinite, then the line \( x = a \) is a vertical asymptote (VA) for \( y = f(x) \).

Ex. 6

Graph \( f(x) = \frac{x}{x^2 - 4} \).

Solution:

\[
\begin{align*}
f'(x) &= \frac{-(x^2 + 4)}{(x^2 - 4)^2}, \\
f''(x) &= \frac{12x(x^2 + 12)}{(x^2 - 4)^3}
\end{align*}
\]

1. Information from \( f(x) \):
**Horizontal Asymptotes**

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x}{x^2 - 4} = \lim_{x \to \infty} \left( \frac{\frac{1}{x^2}}{1 - \frac{4}{x^2}} \right) \\
= \lim_{x \to \infty} \left( \frac{\frac{1}{x} \cdot \frac{1}{1 - 4/x^2}}{1} \right) = 0 \cdot \frac{1}{1-0} = 0 \\
\lim_{x \to -\infty} f(x) = 0 \text{ (same as previous calculation)}
\]

The only HA is \( y = 0 \).

**Vertical Asymptotes**

(look at where denominator is 0.

\( x^2 - 4 = 0 \Rightarrow x = -2 \text{ or } x = 2 \).)

\[
\lim_{x \to -2^-} \frac{x}{x^2 - 4} = \frac{-2}{+\infty} = -\infty \\
\lim_{x \to -2^+} \frac{x}{x^2 - 4} = \frac{-2}{-\infty} = +\infty
\]

D.S. of \( x = -2 \) gives \( -\frac{2}{0} \) for both. So each limit is infinite. Positive or negative?
\[
\lim_{{x \to 2^-}} \frac{x}{{x^2 - 4}} = \frac{2}{0} \quad \infty = -\infty \\
\lim_{{x \to 2^+}} \frac{x}{{x^2 - 4}} = \frac{2}{0} \quad \infty = +\infty \\
\] 

D.S. of \( x = 2 \) gives "\( \frac{2}{0} \)" for both. So each limit is infinite. Positive or negative? So \( x = -2 \) and \( x = 2 \) are VA's.

2) Information from \( f'(x) \):

Recall: \( f'(x) = \frac{-(x^2 + 4)}{(x^2 - 4)^2} \)

First-order critical #'s

- \( f'(x) \) due: none \((x = \pm 2 \) not in domain of \( f \), so not critical #)

- \( f''(x) = 0: -(x^2 + 4) = 0 \)

no solution

New construct sign chart for \( f'(x) \):
(Critical #'s AND vertical asymptotes are
f(x) = \frac{-(x^2 + 4)}{(x^2 - 4)^2}

f'(x) = -\frac{(x^2 + 4)}{(x^2 - 4)^2}

f'(-3) = \frac{-}{\circ} + \circ = \circ

f'(0) = \frac{-}{\circ} + \circ = \circ

f'(3) = \frac{-}{\circ} + \circ = \circ

f is decreasing on (-\infty, -2), (-2, 2), (2, \infty)
f is increasing on no interval
f has local min @ no x-value
f has local max @ no x-value

Information from f''(x):

Recall: f''(x) = \frac{12x(x^2 + 12)}{(x^2 - 4)^3}
Second-order critical #'s:
- \( f''(x) \) due: none
- \( f''(x) = 0 \): \( 12x(x^2+12) = 0 \)
  \[ x = 0 \] only

New construct sign chart for \( f''(x) \):
(Critical #'s AND vertical asymptotes are cut points on sign chart.)

\[ f''(x) = \frac{12x(x^2+12)}{(x^2-4)^3} \]

\( f''(-3) = \frac{\text{--} +}{+} = \text{--} \), \( f''(1) = \frac{++}{-} = - \)

\( f''(-1) = \frac{\text{--} +}{-} = \text{+} \), \( f''(3) = \frac{++}{+} = + \)

\( f \) is concave down on \( (-\infty, -2) \), \( (0, 2) \)
\( f \) is concave up on \( (-2, 0), (2, \infty) \)
\( f \) has inflection point @ \( x = 0 \)
(No inflection point at $x = -2$ or $x = 2$!

The lines $x = \pm 2$ are vertical asymptotes)

4. Graph $y = f(x)$.

- **Important features of Graph:**
  - $(0,0)$: inflection point
  - $y = 0$: horizontal asymptote
  - $x = -2$: vertical asymptote
  - $x = 2$: vertical asymptote

Summary of Info from $f, f', f''$:

<table>
<thead>
<tr>
<th>Inc./dec.</th>
<th>Concave up/down</th>
</tr>
</thead>
<tbody>
<tr>
<td>dec.</td>
<td>Conc. down</td>
</tr>
<tr>
<td>dec.</td>
<td>Conc. up</td>
</tr>
<tr>
<td>dec.</td>
<td>Conc. down</td>
</tr>
</tbody>
</table>

- $\infty$  
- $-2$ 
- $0$ 
- $2$ 
- $\infty$
Section 4.5: L'Hôpital's Rule

**Theorem:** Suppose $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

as long as the limit on the right side exists or is infinite.

**Important Alternatives**

- okay if $x \to a^-$ or if $x \to a^+$.
- okay if $x \to -\infty$ or if $x \to \infty$.
- okay if both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are infinite.

**Indeterminate Forms**

Undefined expressions which do not give enough information to evaluate the limit.

- **Quotients**
  
  $0 \div 0$, $\frac{0}{0}$

  L'Hôpital's Rule (LR) applies directly only to quotients
Products
0 \cdot \infty

Exponents
0^0, 1^\infty, \infty^0

Differences
\infty - \infty, -\infty + \infty

We will ultimately use LR for these forms, but we must do algebra first to transform the expression into a quotient.

Ex. 1

Calculate \( \lim_{x \to 3} \frac{x^2 - 3x}{x^3 - 3x^2 - x + 3} \).

Solution:

D.S. of \( x = 3 \) gives \( \frac{0}{0} \). So we use LR

\[
\lim_{x \to 3} \frac{x^2 - 3x}{x^3 - 3x^2 - x + 3} = \lim_{x \to 3} \frac{2x - 3}{3x^2 - 6x - 1}
\]

not standard notation, indicates where LR has been used.

Now D.S. of \( x = 3 \) gives \( \frac{3}{8} \), so the
limit is $\frac{3}{8}$.

**Ex. 2**

Calculate \( \lim_{x \to 1} \frac{e^x - e}{\ln(x)} \).

**Solution:**

D.S. if \( x = 1 \) gives \( \frac{0}{0} \) \((\ln(1) = 0)\)

So we use LR.

\[
\lim_{x \to 1} \frac{e^x - e}{\ln(x)} = \lim_{x \to 1} \frac{e^x}{1} = \frac{e^1}{1} = e
\]

L'Hospitalize the expression

**Ex. 3**

Calculate \( \lim_{x \to \frac{\pi}{2}^+} \frac{\cos(x)}{1 - \sin(x)} \).

**Solution:**

D.S. of \( x \to \frac{\pi}{2}^+ \) gives \( \frac{0}{0} \). So use LR

\[
\lim_{x \to \frac{\pi}{2}^+} \frac{\cos(x)}{1 - \sin(x)} = \lim_{x \to \frac{\pi}{2}^+} \frac{-\sin(x)}{-\cos(x)} = \lim_{x \to \frac{\pi}{2}^+} \tan(x)
\]
Recall: \( \tan \left( \frac{\pi}{2} \right) \) is undefined.

From the graph of \( y = \tan(x) \),
\[
\lim_{{x \to \frac{\pi}{2}^+}} \tan(x) = -\infty
\]

Alternatively, what if we do not consider \( \tan(x) \)?
\[
\lim_{{x \to \frac{\pi}{2}^+}} \frac{\sin(x)}{\cos(x)} = \frac{0}{0} = -\infty
\]
D.S. gives "\( \frac{1}{0} \)" so limit is infinite as \( x \to \frac{\pi}{2}^+ \), \( \cos(x) \to 0^- \)
Ex. 4

Calculate \( \lim_{x \to 0} \frac{\sin(3x) - 3x + \frac{9x^3}{2}}{x^5} \)

Solution:

D.S. of \( x = 0 \) gives \( \frac{0}{0} \). So use LR.

\[
\lim_{x \to 0} \frac{\sin(3x) - 3x + \frac{9x^3}{2}}{x^5} = \frac{0}{0}
\]

\[
H = \lim_{x \to 0} \frac{3 \cos(3x) - 3 + 27x^2/2}{5x^4} = \frac{0}{0}
\]

\[
H = \lim_{x \to 0} \frac{-9 \sin(3x) + 27}{20x^3} = \frac{0}{0}
\]

\[
H = \lim_{x \to 0} \frac{-27 \cos(3x) + 27}{60x^2} = \frac{0}{0}
\]

\[
H = \lim_{x \to 0} \frac{81 \sin(3x)}{120x} = \frac{0}{0}
\]

\[
H = \lim_{x \to 0} \frac{243 \cos(3x)}{120} = \frac{243}{120}
\]
Calculate \( \lim_{x \to \infty} \frac{\sqrt{x^2+1}}{x} \).

Solution:

\[
\lim_{x \to \infty} \sqrt{x^2+1} = \sqrt{\infty} + 1 = \sqrt{\infty} = \infty
\]

D.S. of “\( x \to \infty \)” gives “\( \frac{\infty}{\infty} \)”\(^1\). So we use LR

\[
\lim_{x \to \infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x \to \infty} \frac{1}{2} (x^2+1)^{-1/2} \cdot (2x)
\]

\[
= \lim_{x \to \infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x \to \infty} \frac{1}{2} (x^2+1)^{-1/2} \cdot (2x)
\]

\[
= \lim_{x \to \infty} \frac{\sqrt{x^2+1}}{x} = \ldots \text{ endless loop}
\]

So LR is 100% useless for this problem. So our old methods are not obsolete! To calculate this limit, see \( \text{Ex. 4} \) in Section 4.4 notes.
\[
\lim_{x \to \infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x \to \infty} \frac{\sqrt{x^2} \cdot \sqrt{1+\frac{1}{x^2}}}{x} = \lim_{x \to \infty} \left( \frac{1}{x} \cdot \sqrt{1+\frac{1}{x^2}} \right)
\]

If \( x \to \infty \), we may assume \( x > 0 \). So \( |x| = x \).

\[
= \lim_{x \to \infty} \sqrt{1+\frac{1}{x^2}} = \sqrt{1+0} = 1
\]

**Ex. 6**

Calculate \( \lim_{x \to 0^+} (x \ln(x)) \).

**Solution:**

\[
\lim_{x \to 0^+} x = 0
\]

\[
\lim_{x \to 0^+} \ln(x) = -\infty
\]

\[
y = \ln(x)
\]
D.S. of “$x \to 0^+$” gives “$0 \cdot (-\infty)$”, which is indeterminate.

Note: We cannot use LR directly because “$x \ln(x)$” is not a quotient. Don\'t make up your own rule.

- Rewrite as quotient using algebra
- Then use LR!

$$\lim_{x \to 0^+} (x \ln(x)) = \lim_{x \to 0^+} \frac{\ln(x)}{1/x} = \lim_{x \to 0^+} \frac{\ln(x)}{x^{-1}}$$

So LR can be used $\implies$ “$-\infty \over \infty$”

$$H = \lim_{x \to 0^+} \frac{1/x}{-x^{-2}} = \lim_{x \to 0^+} (-x) = 0$$

---

**Ex. 7**

Let $f(x) = x^2 e^{-x}$. Find all horizontal asymptotes of $f(x)$.

**Solution:**
To find the HA's, we must compute

(a) \( \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^2 e^{-x} \)

(b) \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} x^2 e^{-x} \).

Recall: Limits involving \( e^x \).

\[
\lim_{x \to -\infty} e^x = 0 \\
\lim_{x \to \infty} e^x = \infty \\
\lim_{x \to -\infty} e^{-x} = \infty \\
\lim_{x \to \infty} e^{-x} = 0
\]

(a) \[
\begin{align*}
\lim_{x \to -\infty} x^2 e^{-x} &= \infty \\
\lim_{x \to -\infty} x^2 &= \infty \\
\lim_{x \to -\infty} e^{-x} &= \infty
\end{align*}
\]

D.S. of “\( x \to -\infty \)” gives “\( \infty \cdot \infty \)”, which is not indeterminate.
\[
\lim_{x \to -\infty} x^2 e^{-x} = \infty
\]
So no HA in direction \( x \to -\infty \).

(b) \[
\lim_{x \to \infty} x^2 e^{-x}
\]
\[
\lim_{x \to \infty} x^2 = \infty , \quad \lim_{x \to \infty} e^{-x} = 0
\]
D.S. of \( "x \to \infty" \) gives \( "\infty \cdot 0" \).
\[
\lim_{x \to \infty} (x^2 e^{-x}) = \lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x}
\]
\[
\text{"}\frac{\infty}{\infty}\text{"} \quad \text{"}\frac{\infty}{\infty}\text{"}
\]
\[
\text{H} = \lim_{x \to \infty} \frac{2}{e^x} = 0
\]
\( \text{"}\frac{2}{\infty}\text{"} \) not indeterminate

So the only HA of \( f(x) \) is \( y = 0 \).
Ex. 8

Calculate \( \lim_{x \to 0^+} (x^x) \).

**Solution:**

D.S. of “\( x \to 0^+ \)” gives “\( 0^0 \)”, which is an indeterminate exponent.

**Q:** How do you write \( x^x \) as a quotient?

**A:** Use logarithms!

- Let \( L \) be the desired limit.
- Consider \( \ln (L) \) instead.
- Use algebra and log-rules to write as quotient.
- Then use LR!
- Solve for \( L \)

Put \( L = \lim_{x \to 0^+} (x^x) \).

\[
\ln (L) = \ln \left[ \lim_{x \to 0^+} (x^x) \right] = \lim_{x \to 0^+} \ln (x^x)
\]
Since \( \ln(x) \) is continuous on its domain, "ln" and "lim" can be swapped.

\[
\lim_{x \to 0^+} (x \ln(x)) = \lim_{x \to 0^+} \left( \frac{\ln(x)}{1/x} \right)
\]

\[
= -\infty
\]

\[
H = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.
\]

So \( \ln(L) = 0 \), whence \( L = 1 \).

**Ex. 9**

Calculate \( \lim_{x \to 0} (\cos(x))^{3/x^2} \).

**Solution:**

\[
\lim_{x \to 0} \cos(x) = 1, \quad \lim_{x \to 0} \frac{3}{x^2} = \infty
\]

D.S. of "\( x \to 0 \)" gives "\( 1^{\infty} \)", which is indeterminate. Put \( L = \lim_{x \to 0} (\cos(x))^{3/x^2} \).
Consider \( \ln(L) \).

\[
\ln(L) = \lim_{x \to 0} \ln \left[ (\cos(x))^\frac{3}{x^2} \right]
\]

\[
= \lim_{x \to 0} \left[ \frac{3}{x^2} \cdot \ln(\cos(x)) \right] = \lim_{x \to 0} \left[ \frac{3 \ln(\cos(x))}{x^2} \right]
\]

"\( \frac{0}{0} \)" use LR!

\[
H = \lim_{x \to 0} \left[ 3 \cdot \frac{1}{\cos(x)} \left( -\sin(x) \right) \right]
\]

\[
= \lim_{x \to 0} \left[ -\frac{3 \tan(x)}{2x} \right] = \lim_{x \to 0} \left[ \frac{-3 \sec(x)^2}{2} \right] = -\frac{3}{2}
\]

"\( \frac{0}{0} \)"

So \( \ln(L) = -\frac{3}{2} \), whence \( L = e^{-\frac{3}{2}} \).
Calculate \( \lim_{x \to 0^-} \left( \frac{1}{\sin(x)} - \frac{1}{x} \right) \)

**Solution:**

\[
\begin{align*}
\lim_{x \to 0^-} \frac{1}{\sin(x)} &= \frac{\infty}{0} \quad \text{as} \quad x \to 0^- \\
\lim_{x \to 0^-} \frac{1}{x} &= \frac{\infty}{0} \quad \text{as} \quad x \to 0^- \\
\end{align*}
\]

D.S. of “\( x \to 0^- \)” gives “\( -\infty - (-\infty) \),” or “\( -\infty + \infty \),” which is indeterminate.

**Goal:** write difference as a quotient. Then use L.R. Common denominator.

\[
\begin{align*}
\lim_{x \to 0^-} \left( \frac{1}{\sin(x)} - \frac{1}{x} \right) &= \lim_{x \to 0^-} \left( \frac{x - \sin(x)}{x \sin(x)} \right) \quad \text{“}\frac{0}{0}\text{”}
\end{align*}
\]

\[
H = \lim_{x \to 0^-} \frac{1 - \cos(x)}{x \cos(x) + \sin(x)} \quad \text{“}\frac{0}{0}\text{”}
\]
\[
\lim_{x \to 0^-} \frac{\sin(x)}{-x \sin(x) + \cos(x) + \cos(x)} = \frac{0}{0 + 1 + 1} = 0
\]
The difference of two numbers is 10. Find their minimum product.

Solution:
Let $x$ and $y$ be the two numbers. We want to minimize the function

$$p(x, y) = xy$$

Problem: This is a function of more than one variable (uh-oh!).

However, $x$ and $y$ are not independent of each other. Instead they satisfy the constraint equation

$$x - y = 10$$

We use the constraint to write the objective in terms of one variable.
\[ x = y + 10 \]

\[ \downarrow \text{Substitute into objective function} \]

Our objective function is now

\[ p(y+10, y) = (y+10)y = y^2 + 10y \]

**Goal**: Find the absolute minimum value of the function

\[ f(y) = y^2 + 10y \]

on the interval \((-\infty, \infty)\).

\[ \downarrow \text{In Math 111/115, this is a valid problem up until this point.} \]

Find the critical \#s:

- \( f'(y) \) does: none
- \( f'(y) = 0 \):
  
  \[ f'(y) = 2y + 10 = 0 \]
  
  \[ \implies y = -5 \]

Since \((-\infty, \infty)\) is not bounded, we
Cannot use the method of Section 4.1 to determine the absolute minimum. Worse, the minimum may not even exist. So we will use methods of Sections 4.3 and 4.4.

What does $f'$ tell us? Construct a sign chart for $f'$.

![Sign Chart](image)

Since $f$ is decreasing on $(-\infty, -5]$ and increasing on $[-5, \infty)$, the absolute min of $f$ occurs at $y = -5$. So the minimum product is $f(-5) = -25$. 

\[
f'(y) = 2y + 10
\begin{align*}
f'(-6) &= -2 < 0 \\
f'(0) &= 10 > 0
\end{align*}
\]
A cylindrical tank has volume $2000\pi$ m$^3$. Find the dimensions of the tank with the smallest possible surface area.

**Hint:** $A = 2\pi r^2 + 2\pi rh$

**Solution:**
We want to minimize the function

$$A(r, h) = 2\pi r^2 + 2\pi rh$$

subject to the constraint equation

$$2000\pi = \pi r^2 h$$

The only role of this equation is
to let us write $A(r, h)$ in terms of one variable only. We never differentiate the constraint equation.

Solving for $h$ in terms of $r$:

$$h = \frac{2000}{r^2}$$

Substituting into the objective gives

$$A(r, \frac{2000}{r^2}) = 2\pi r^2 + 2\pi r \left( \frac{2000}{r^2} \right)$$

$$= 2\pi \left( r^2 + \frac{2000}{r} \right)$$

Goal: Find the value of $r$ that gives absolute minimum value of the function

$$f(r) = 2\pi \left( r^2 + \frac{2000}{r} \right)$$

on the interval $(0, \infty)$

Find the critical #'s:

- $f'(r)$ does: none (c.f., Section 4.2)
\[ f'(r) = 0: \]
\[ f'(r) = 2\pi \left( 2r - \frac{2000}{r^2} \right) = 0 \]

\[ 2r^3 = 2000 \]
\[ r^3 = 1000 \]
\[ r = 10 \]

Now determine the absolute minimum.

1. **Method 1:** using first derivative

   \[ f'(r) = 2\pi \left( 2r - \frac{2000}{r^2} \right) \]
   \[ f'(1) = 2\pi \left( 2 - \frac{2000}{1} \right) < 0 \]
   \[ f'(1000) = 2\pi \left( \frac{2000}{1000} - \frac{2000}{1000000} \right) > 0 \]

Since \( f \) is decreasing on \((0, 10]\) and increasing on \([10, \infty)\), absolute min of \( f \) occurs at \( r = 10 \).
2. **Method 2:** using second derivative

\[ f''(r) = 2\pi \left( 2r - \frac{2000}{r^2} \right) \]

\[ f''(r) = 2\pi \left( 2 + \frac{4000}{r^3} \right) \]

\[ > 0 \quad > 0 \quad > 0 \quad \text{if } r > 0 \]

So \( f''(r) > 0 \) for all \( r > 0 \). So \( f \) is concave up on \((0, \infty)\). Since \( r = 10 \) is the only critical #, absolute min of \( f \) occurs at \( r = 10 \).

**Final Answer:** \( r = 10, \quad h = \frac{2000}{10^2} = 20 \)

---

**Ex. 3**

A rectangle has its lower left vertex at the origin and its upper right vertex on the graph of \( y = \frac{4-x}{2+x} \). Find the largest possible area of such a rectangle.
Solution:

Let w and h be the width and height of the rectangle, respectively. We want to maximize the function

\[ A(w, h) = wh \]

subject to the constraint equation

\[ h = \frac{4-w}{2+w} \]
Substituting into the objective gives

\[ A(w, \frac{4-w}{2+w}) = w \cdot \frac{4-w}{2+w} = \frac{4w - w^2}{2+w} \]

**Goal:** Find the absolute maximum area of the function

\[ f(w) = \frac{4w - w^2}{2+w} \]

on the interval \([0, 4]\).

Find the critical ***s:***

- \(f'(w)\) due: none \(\text{(c.f. Section 4.2)}\)
- \(f''(w) = 0: \)

\[ f'(w) = \frac{(2+w)(4-2w) - (4w-w^2)(1)}{(2+w)^2} \]

\[ f'(w) = \frac{8-4w-w^2}{(2+w)^2} = 0 \]

\[ 8-4w-w^2 = 0 \]

\[ w = -2 - 2\sqrt{3} \quad \text{or} \quad w = -2 + 2\sqrt{3} \]
not in $[0,4]$ (it's negative!)

Now determine the absolute max of $f$. Since $[0,4]$ is closed and bounded, we can use the method of Section 4.1.

$$f(w) = \frac{w(4-w)}{2+w}$$

<table>
<thead>
<tr>
<th>$w$-values</th>
<th>$y$-values</th>
</tr>
</thead>
<tbody>
<tr>
<td>endpt 0</td>
<td>0</td>
</tr>
<tr>
<td>endpt 4</td>
<td>0</td>
</tr>
<tr>
<td>critical $-2+2\sqrt{3} \approx 1.4$</td>
<td>$\approx \frac{(1.4)(2.6)}{3.4} &gt; 0$</td>
</tr>
</tbody>
</table>

So the absolute maximum of $f$ is

$$f(-2+2\sqrt{3}) = \frac{(-2+2\sqrt{3})(6-2\sqrt{3})}{2\sqrt{3}} = \frac{8-4\sqrt{3}}{2\sqrt{3}}$$

**Ex. 4**

A rectangular box has total surface area $450 \text{ m}^2$, and the length is three
times its width.
(a) Find \( f(w) \), the volume of the box in terms of the width \( w \). What is the domain of \( f \) in the context of this problem?
(b) Find the dimensions of such a box with the largest possible volume.

\textbf{Solution:}

(a) Let \( l, w, h \) be the length, width and height of the box, respectively. We want to maximize the function

\[ V(l, w, h) = lwh \]

subject to the constraint equations

\[ \begin{align*}
(1) \quad & l = 3w \\
(2) \quad & 450 = 2lw + 2lh + 2wh
\end{align*} \]
Put (1) into (2):

\[(2)^* \quad 450 = 6w^2 + 6wh + 2wh\]

Now solve for \( h \) in terms of \( w \).

\[450 = 6w^2 + 8wh\]

\[\Rightarrow h = \frac{450 - 6w^2}{8w} = \frac{225 - 3w^2}{4w}\]

Now rewrite objective in terms of \( w \).

\[V(3w, w, \frac{225 - 3w^2}{4w}) = 3w \cdot w \cdot \frac{225 - 3w^2}{4w}\]

\[f(w) = \frac{3}{4} \left( 225w - 3w^3 \right) \]  
\text{volume function}

What are the allowed values of \( w \)?
Since all length are \( \geq 0 \), we must have
The domain of \( f(w) \) is \((0, \sqrt{75}]\).

**Goal**: Find the value of \( w \) that gives the absolute maximum value of the function
\[
f(w) = \frac{3}{4} (225w - 3w^3)
\] on the interval \((0, \sqrt{75}]\).

**(b) Find the critical #s.**

- \( f'(w) \) dne: none
- \( f'(w) = 0 \):
  \[
f'(w) = \frac{3}{4} (225 - 9w^2) = 0
  \]
  \[
  9w^2 = 225 \implies w^2 = 25 \implies w = 5
  \]
Determine absolute maximum of $f$. Use the second derivative.

$$f''(w) = \frac{3}{4} (-18w) = -\frac{27}{2} w$$

Observe that $f''(w) < 0$ for $w$ in $(0, \sqrt{75}]$. So $f$ is concave down on $(0, \sqrt{75}]$. Since $w = 5$ is the only critical #, $w = 5$ gives absolute max. So the dimensions of the box with the largest volume are $l = 15$, $w = 5$, and $h = 7.5$.

**Ex. 5**

Find the equation of the line through $P = (12, 4)$ such that the triangle bounded by the line and coordinate axes has minimum area.

Solution:
Any line that passes through P must have an equation of the form

\[ y = 4 + m(x - 12) \]

where \( m \) is an unknown slope. So our variable will be the slope \( m \).

Find width and height of triangle in terms of \( m \):

Width: \((w,0)\) lies on line, so...

\[ 0 = 4 + m(w - 12) \implies w = -\frac{4}{m} + 12 \]

formula for \( w \) in terms of \( m \)
height: \((0,h)\) lies on line, so...
\[ h = 4 + m(0-12) \Rightarrow h = 4 - 12m \]

So the area of the triangle is
\[ f(m) = \frac{1}{2} \left( -\frac{4}{m} + 12 \right) \left( 4 - 12m \right) \]

area of \( \Delta = \frac{1}{2} \) wh

Note: The line must have a negative slope!

Goal: Find value of \( m \) that gives the minimum value of function
\[ f(m) = 48 - \frac{8}{m} - 7.2m \]
on the interval \((-\infty, 0)\).

Find critical #1s:
- \( f'(m) \) dne: nowhere
Now determine the absolute min of f. Use the second derivative!

\[ f''(m) = \frac{-16}{m^3} \]

Observe that \( f''(m) > 0 \) if \( m < 0 \), i.e., if \( m \) is in \((-\infty, 0)\). So \( f \) is concave up on \((-\infty, 0)\). Since \( m = -\frac{1}{3} \) is the only critical \( \# \), \( f \) must have an absolute minimum at \( m = -\frac{1}{3} \). The equation of the line is thus

\[ y = 4 - \frac{1}{3} (x-12) \]
A differentiable function is "locally linear." In other words, the tangent line to \( f(x) \) at \( x = a \) is an approximation of \( f(x) \) near \( x = a \).

**Analytical Intuition**

By the definition of derivative,

\[
f'(a) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}
\]

If \( \Delta x \) is small (\( \Delta x \) is near 0), then \( f'(a) \approx \frac{\Delta f}{\Delta x} \).

So if \( \Delta x \) is small...
\[
\begin{align*}
    f'(a) &\approx \frac{f(a + \Delta x) - f(a)}{\Delta x} \\
    f(a + \Delta x) &\approx f(a) + f'(a) \Delta x \\
    y &\approx f(a) + f'(a) (x-a)
\end{align*}
\]

**Tangent line approximation**

---

**Terminology and Notation**

1. **Linearization**
   - synonymous with tangent line
   - linearization is a function of \( x \).
   - 
     \[ex: \text{Find tangent line to } y = x^2 \text{ at } x = 3.\]
     \[\text{Solution: } y - 9 = 6 (x-3)\]
     \[ex: \text{Find linearization to } y = x^2 \text{ at } x = 3.\]
     \[\text{Solution: } L(x) = 9 + 6 (x-3)\]

2. **Differentials** (more in Chapter 5)
   - right now this is alternative notation
for linear approximation

* If \( y = f(x) \), then the differential of \( y \) at \( x = a \) is \( dy = f'(a) \, dx \)

Ex: Find differential of \( y = x^2 \) at \( x = 3 \).

**Solution:** \( dy = 6 \, dx \) \( (f'(x) = 2x, \, f'(3) = 6) \)

“Small change in \( y \)” “Small change in \( x \)”

If you are at \( x = 3 \) already, an increase of \( x \) by \( dx \) gives an increase in \( y = x^2 \) by about \( dy \).

Ex. 1

Use a linear approximation to estimate the value of \( \tan \left( \frac{\pi}{4} + 0.01 \right) \).

**Solution:**

The phrase “linear approximation” means to use a tangent line approximation. Put \( f(x) = \tan(x) \) and find tangent line to \( f(x) \) at \( x = \pi/4 \).
\[ f(\pi/4) = 1 \]
\[ f'(x) = \sec(x)^2 \]
\[ f'(\pi/4) = 2 \]
So the linearization of \( f(x) \) at \( x = \frac{\pi}{4} \) is:
\[ L(x) = 1 + 2 \left( x - \frac{\pi}{4} \right) \]

Note: If \( x \) is near \( \frac{\pi}{4} \), then
\[ \tan(x) \approx 1 + 2 \left( x - \frac{\pi}{4} \right) \]
Since \( \frac{\pi}{4} + 0.01 \) is near \( \frac{\pi}{4} \),
\[ \tan \left( \frac{\pi}{4} + 0.01 \right) \approx 1 + 2 \left( \frac{\pi}{4} + 0.01 - \frac{\pi}{4} \right) = 1.02 \]
Put \( x = \frac{\pi}{4} + 0.01 \) in highlighted formula

Ex. 2

Use a linear approximation to estimate the value of \( 18^{1/4} \).

Solution:
Put $f(x) = x^{1/4}$ and find tangent line to $f(x)$ at $x = 16$.

\[
f(16) = 16^{1/4} = 2
\]
\[
f'(x) = \frac{1}{4} x^{-3/4}
\]
\[
f''(16) = \frac{1}{4} (16^{1/4})^{-3} = \frac{1}{4} \cdot 2^{-3} = \frac{1}{32}
\]

So the linearization of $f(x)$ at $x = 16$ is

\[
L(x) = 2 + \frac{1}{32} (x-16)
\]

Note: If $x$ is near 16, then

\[
x^{1/4} \approx 2 + \frac{1}{32} (x-16)
\]

Since 18 is near 16,

\[
18^{1/4} \approx 2 + \frac{1}{32} (18-16) = 2 + \frac{1}{16} = 2.0625
\]

**Ex. 3**

**Use a linear approximation to estimate the value of $\sqrt{3.9}$.

**Solution:**
Put $f(x) = \sqrt{x}$ and find tangent line to $f(x)$ at $x = 4$.

- $f(4) = 2$
- $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$
- $f'(4) = \frac{1}{2} \cdot 4^{-\frac{1}{2}} = \frac{1}{4}$

So our linearization to $f(x)$ at $x=4$ is

$$L(x) = 2 + \frac{1}{4}(x-4)$$

**Note:** If $x$ is near 4, then

$$\sqrt{x} \approx 2 + \frac{1}{4}(x-4)$$

Since 3.9 is near 4,

$$\sqrt{3.9} \approx 2 + \frac{1}{4}(3.9-4) = 2 - \frac{1}{40} = 1.975$$

---

**Terminology in Business & Economics**

$x$: # of units sold/produced

$\rightarrow$ unless stated otherwise, we assume entire inventory is sold.
\( p(x) \): price per unit if \( x \) units are sold

\( \rightarrow \) "demand function"

\( R(x) \): total revenue from selling \( x \) units

\( \rightarrow R(x) = \left( \# \text{ of units sold} \right) \cdot \left( \text{price per unit} \right) \)

\( R(x) = xp(x) \)

\( C(x) \): total cost of producing \( x \) units

\( \rightarrow C(0) \): “sunk cost” or “fixed cost”

**Marginal Quantities**

\( MQ(x) \): additional amount of “Q” achieved if \( 1 \) more unit is produced/sold, if \( x \) units are already being produced/sold

\( \rightarrow MQ(x) = Q(x+1) - Q(x) \)

\( MR(x) \): marginal revenue
Additional from the \((x+1)\)th unit OR revenue from 1 more unit if \(x\) units are already being sold.

\(MC(x)\): marginal cost

Additional cost from the \((x+1)\)th unit OR cost from 1 more unit if \(x\) units are already being produced.

**Example:** The cost of producing the 11th unit:

\[ MC(10) = C(11) - C(10) \]

**Standard Approximation of \(MQ(x)\)**

The linearization of \(Q(x)\) at \(x = a\) is

\[ L(x) = Q(a) + Q'(a)(x - a) \]

**Note:** If \(x\) is near \(a\), then

\[ Q(x) \approx Q(a) + Q'(a)(x - a) \]

Since \(a+1\) is near \(a\),

\[ Q(a+1) \approx Q(a) + Q'(a)(a+1-a) = 1 \]
\[ Q(a+1) - Q(a) \approx Q'(a) \]

\[ = MQ(a) \]

**Marginal revenue:**

\[ MR(x) = R(x+1) - R(x) \quad \text{exact} \]

\[ MR(x) \approx R'(x) \quad \text{approximate} \]

**Marginal cost:**

\[ MC(x) = C(x+1) - C(x) \quad \text{exact} \]

\[ MC(x) \approx C'(x) \quad \text{approximate} \]

**Ex. 4**

If a factory produces \( x \) units, then the total cost is

\[ C(x) = \frac{1}{8} x^2 + 3x + 98 \]

and the selling price per unit is

\[ p(x) = 25 - \frac{1}{3} x \]

(a) What is the exact marginal cost function? What is the exact cost of
the 9th unit? Same for revenue?

(b) Use marginal analysis to estimate the cost of the 9th unit. Same for revenue.

Solution:

(a) By definition,

\[
MC(x) = C(x+1) - C(x) \\
= \left[\frac{1}{8}(x+1)^2 + 3(x+1) + 98\right] - \left[\frac{1}{8}x^2 + 3x + 98\right] \\
= \frac{1}{4}x + 3 + \frac{1}{8}
\]

The cost of producing the 9th unit is

\[
MC(8) = \left(\frac{1}{4}x + 3 + \frac{1}{8}\right)\bigg|_{x=8} = 5 + \frac{1}{8}
\]

By definition,

\[
R(x) = xp(x) = 25x - \frac{1}{3}x^2
\]

\[
MR(x) = R(x+1) - R(x)
\]
\[ = \left[ 25(x+1) - \frac{2}{3}(x+1)^2 \right] - \left[ 25x - \frac{1}{3}x^2 \right] \\
= 25 - \frac{2}{3}x - \frac{1}{3} \]

The revenue from the 9th unit is

\[
MR(8) = \left( 25 - \frac{2}{3}x - \frac{1}{3} \right) \bigg|_{x=8} = \frac{59}{3} - \frac{1}{3} 
\]

(b) We use the approximation

\[
MC(x) \approx C'(x) = \frac{1}{4}x + 3 
\]

Recall: \( C(x) = \frac{1}{8}x^2 + 3x + 9.8 \)

\[
MR(x) \approx R'(x) = 25 - \frac{2}{3}x 
\]

Recall: \( R(x) = 25x - \frac{1}{3}x^2 \)

So the approximate cost and approx. revenue from the 9th unit are

\[
MC(8) \approx C'(8) = \left( \frac{1}{4}x + 3 \right) \bigg|_{x=8} = 5 \\
MR(8) \approx R'(8) = \left( 25 - \frac{2}{3}x \right) \bigg|_{x=8} = \frac{59}{3} 
\]
Suppose we measure some quantity to have a value of $x_0$, with a max uncertainty of $\Delta x$. Now we use $x_0$ to calculate the derived quantity $Q(x_0)$. What is the (approximate) max uncertainty in $Q(x_0)$?

measured value of $x$: $x_0$
max uncertainty in $x_0$: $\Delta x$
derived value of $Q$: $Q(x_0)$

(approximate) max uncertainty in $Q(x_0)$: $\Delta Q \approx |Q'(x_0)| \Delta x$

This comes from linear approximation $\Delta Q \approx Q(x_0 + \Delta x) - Q(x_0) \approx Q'(x_0) \Delta x$

Ex. 5: Suppose the radius of a disk is measured to be 2m with max uncertainty of 1cm (0.01m). This measurement is
used to calculate the area of the disk.

(a) What is the approximate maximum uncertainty in the area?

(b) What is the approximate maximum percentage error in the area?

Solution:

(a) In the notation of the preamble,

\[ x_0 = 2 \]
\[ \Delta x = 0.01 \]

\[ Q(x) = \pi x^2 \]

→ area of disk, given radius is \( x \)

\[ Q'(x) = 2\pi x \]

\[ Q'(x_0) = 2\pi \cdot 2 = 4\pi \]

So the max error in the area is approximately

\[ \Delta Q \approx | Q'(x_0) \Delta x | = | 4\pi \cdot (0.01) | = 0.04\pi \]

(b) In general, the percentage error is
\[ \% \text{error} = \frac{\text{absolute error}}{\text{measurement}} \times 100 \]

So our max \% error is about

\[ \% \text{error} \approx \frac{\Delta Q}{Q(x_0)} \times 100 = \frac{0.04\pi}{\pi (2)^2} \times 100 \]

\[ \% \text{error} \approx \frac{0.04\pi}{4\pi} \times 100 = \frac{4\pi}{4\pi} = 17\% \]
Section 4.7: Optimization in Econ / Business

Maximizing Profit

\( \pi(x) : \) total profit = (total rev.) - (total cost)
\( \Rightarrow P(x) \)
\( \pi(x) = R(x) - C(x) \)

To maximize profit, find the critical \#'s of \( \pi(x) \).

\[ \pi'(x) = R'(x) - C'(x) = 0 \]

\[ R'(x) = C'(x) \quad \text{exact} \]

\[ MR(x) = MC(x) \]

(\text{This uses the linear approximation})
\[ MR(x) \approx R'(x) \quad \text{and} \quad MC(x) \approx C'(x) \]

* Economic theory tells us that \( \pi(x) \) has a unique local maximum. So if we use the "MR = MC" formalism then there is no need to verify the maximum of \( \pi(x) \) explicitly.

\[ \text{Ex. 1} \]

Suppose total cost and market price are
given by
\[ C(x) = \frac{2}{5} x^2 + 4x + 44 \]
\[ p(x) = \frac{1}{5} (62 - x) \]

Find the optimal level of production.

Solution:

Unless otherwise stated, "optimal level of production" always means "find value of \( x \) that maximizes total profit".

So use "\( MR = MC \)".

Revenue: \( R(x) = x \cdot p(x) = \frac{1}{5} (62x - x^2) \)

Cost: \( C(x) = \frac{2}{5} x^2 + 4x + 44 \)

\[ MR = MC \quad (R' = C') \]

\[ \frac{1}{5} (62 - 2x) = \frac{4}{5} x + 4 \]

\( \Downarrow \quad x = 7 \)

We do not need to verify that \( x = 7 \) gives max profit since we used the
"MR = MC" formalism. So profit is maximized when $x = 7$.

Ex. 2

Given that the total cost of producing $x$ widgets is

$$C(x) = 3x^2 + x + 48 \ldots$$

(a) Find the level of production that minimizes average cost per unit.
(b) Find exact cost of producing 21st unit.
(c) Using marginal analysis, estimate the cost of producing the 21st unit.

Solution:

(a) By definition, the average cost is

$$AC(x) = \frac{C(x)}{x} = 3x + 1 + \frac{48}{x}$$

(We want to find the value of $x$ that gives absolute minimum value of $AC(x)$)
on the interval \((0, \infty)\).

Find the critical \#s:

- \(AC'(x)\) dne: none
- \(AC'(x) = 0\):

\[
AC'(x) = 3 - \frac{48}{x^2} = 0
\]

\(\Rightarrow x = -4\) or \(x = 4\)

not in \((0, \infty)\)

This is not an "MR = MC" problem, so we have to verify whether \(x = 4\) gives the absolute minimum.

Observe that

\[
AC''(x) = \frac{96}{x^3}
\]

is positive for \(x > 0\). So \(AC(x)\) is concave up on \((0, \infty)\). Hence \(x = 4\), being the only critical number, must give the absolute minimum of \(AC(x)\).
(b) Recall: \( C(x) = 3x^2 + x + 48 \)

The exact cost of the 21st unit is
\[
C(21) - C(20) = 3(21^2 - 20^2) + (21-20) + (48-48)
= 3(21-20)(21+20) + 1 + 0
= 3(1)(41) + 1 + 0 = 123 + 1 = 124
\]

(c) The approximate cost of the 21st unit is
\[
C'(20) = 6 \cdot 20 + 1 = 121 \quad C'(x) = 6x + 1
\]

Recall from Section 3.8, \( C(x+1) - C(x) \approx C'(x) \)

---

**Ex. 3**

The manufacturing cost of producing \( x \) widgets is
\[
C(x) = \frac{1}{10}x + 6 \quad (\text{in dollars})
\]

and the market price is
\[
p(x) = \frac{70 - x}{30 + x}
\]
In addition, the manufacturer must pay a government tax of $0.10 per unit produced. What is the optimal level of production?

**Solution:**

We will use "MR = MC" to maximize total profit.

**Revenue:** 
\[ R(x) = x \cdot p(x) = \frac{70x - x^2}{30 + x} \]

**Cost:** 
\[ \tilde{C}(x) = C(x) + T(x) \]

- **Total cost:** \( C(x) \)
- **Manufacturing cost:** \( C(x) \)
- **Tax cost:** \( T(x) \)

\[ \tilde{C}(x) = \frac{1}{10}x + 6 + \frac{1}{10}x \]

So now use \( MR = MC \).

\[ MR = MC \quad (R' = \tilde{C}') \]
\[
\frac{(30+x)(70-2x)-(70x-x^2)}{(30+x)^2} = \frac{1}{5}
\]

\[
\frac{-x^2 - 60x + 2100}{(30+x)^2} = \frac{1}{5}
\]

\[-x^2 - 60x + 2100 = \frac{1}{5} (900 + 60x + x^2)\]

\[0 = \frac{6}{5} x^2 + 72x - 1920\]

\[0 = \frac{6}{5} (x+80)(x-20)\]

\[\Rightarrow x = -80 \quad \text{or} \quad x = 20\]

*can't have negative level of production*

Since we used "MR = MC" formalism, no need to verify the maximum; \(x = 20\) gives maximum profit.
Ex. 1

Consider \( f \) below.

\[
f(x) = 1 - \frac{9}{x} + \frac{6}{x^2}
\]

Find all of the following for the graph of \( y = f(x) \).

(a) vertical asymptotes
(b) horizontal asymptotes
(c) where \( f \) is decreasing
(d) where \( f \) is increasing
(e) local min/max
(f) where \( f \) is concave down
(g) where \( f \) is concave up
(h) inflection points
(i) sketch graph

Solution:
First find \( f' \) and \( f'' \).
\begin{align*}
\text{f}(x) &= 1 - \frac{9}{x} + \frac{6}{x^3} = \frac{x^3-9x^2+6}{x^3} \\
\text{f}(x) &= \frac{9}{x^2} - \frac{18}{x^4} = \frac{9(x^2-2)}{x^4} \\
\text{f}''(x) &= -\frac{18}{x^3} + \frac{72}{x^5} = \frac{18(4-x^2)}{x^5}
\end{align*}

**Information from f:**

(a) vertical asymptotes  
(b) horizontal asymptotes

- VA: \( x = 0 \)

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x^3-9x^2+6}{x^3} = \frac{+6}{-\infty} = -\infty \\
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x^3-9x^2+6}{x^3} = \frac{+6}{+\infty} = +\infty
\]

Recall: From Sec. 4.4, the form "\( \frac{\text{nonzero}}{0} \)" means the one-sided limits are infinite.

- HA: We have to calculate both limits
at \( \infty \):
\[
\lim_{x \to \infty} \left( \frac{x^3 - 9x^2 + 6}{x^3} \right) = \lim_{x \to \infty} \left( 1 - \frac{9}{x} + \frac{6}{x^3} \right)
\]
\[
= 1 - 0 + 0 = 1 \quad (\text{can also use LR})
\]
\[
\lim_{x \to -\infty} \left( \frac{x^3 - 9x^2 + 6}{x^3} \right) = 1 \quad (\text{similarly})
\]

So the HA is \( y = 1 \).

Information from \( f' \):
- intervals of decrease
- intervals of increase
- local min/max

First-order critical #s:
- \( f'(x) \) dne: none \( (x=0 \) is not a critical # since \( x=0 \) is not in the domain of \( f \))
- \( f'(x) = 0 \): \( x = \sqrt{2}, \ x = -\sqrt{2} \)

Construct a sign chart (number line) for \( f' \):
\[ f'(x) = \frac{9(x^2-2)}{x^4} \]

\[ f'(-2) = \frac{+}{+} = + \quad f'(1) = \frac{+}{-} = - \]

\[ f'(-1) = \frac{+}{-} = - \quad f'(2) = \frac{+}{+} = + \]

\( f \) is decreasing on \([-\sqrt{2}, 0), (0, \sqrt{2}]\)
\( f \) is increasing on \((-\infty, -\sqrt{2}], [\sqrt{2}, \infty)\)

Local min @ \( x = \sqrt{2} \)
Local max @ \( x = -\sqrt{2} \)

Information from \( f'' \):
(f) concave down
(g) concave up
(h) Inflection points

\[ f''(x) = \frac{18(4-x^2)}{x^5} \]
Second-order critical #’s:

• $f''(x)$ dne: none

• $f''(x) = 0$: $x = -2, \ x = 2$

Construct a sign chart for $f''$:

\[ \begin{array}{cccccc}
& -3 & -1 & 0 & 1 & 3 \\
\uparrow & \uparrow & \downarrow & \uparrow & \uparrow \\
+ & -2 & - & + & + \\
\end{array} \]

Shape of $f$

Sign of $f''$

Test point

\[ f''(x) = \frac{18(4-x^2)}{x^5} \]

\[ f''(-3) = \frac{+}{-} = + \quad f''(1) = \frac{+}{+} = + \]

\[ f''(-1) = \frac{+}{-} = - \quad f''(3) = \frac{+}{+} = + \]

$f$ is concave down: $[-2, 0)$, $[2, \infty)$

$f$ is concave up: $(-\infty, -2]$, $(0, 2)$

Inflection points @ $x = -2, 2$

[Sketch Graph of $f(x)$]
\[ f(x) = 1 - \frac{9}{x} + \frac{6}{x^3} = 1 + \frac{6 - 9x^2}{x^3} \]

\[ f(-\sqrt{2}) = 1 + \frac{6 - 9(2)}{-2\sqrt{2}} = 1 + \frac{12}{2\sqrt{2}} \]

\[ f(\sqrt{2}) = 1 + \frac{6 - 9(2)}{2\sqrt{2}} = 1 - \frac{12}{2\sqrt{2}} \]

\[ f(-2) = 1 + \frac{6 - 9(4)}{-8} = 1 + \frac{30}{8} \]
Calculate the limit or show it DNE.

\[ \lim_{x \to 1} \left( \frac{xe^{4x} + 4e^4 - 5e^{4x}}{(x - 1)^2} \right) \]

**Solution:**

D.S. of \( x = 1 \) gives \( \frac{0}{0} \). So use LR.

\[ \lim_{x \to 1} \left( \frac{xe^{4x} + 4e^4 - 5e^{4x}}{(x - 1)^2} \right) \]

\[ = \lim_{x \to 1} \left( \frac{xe^{4x} \cdot 4 + e^{4x} \cdot 1 + 0 - 5e^4}{2(x - 1)} \right) \]

\[ = \lim_{x \to 1} \left( \frac{4xe^{4x} + e^{4x} - 5e^4}{2(x - 1)} \right) \]

\[ = \lim_{x \to 1} \left( \frac{4xe^{4x} + e^{4x} - 5e^4}{2(x - 1)} \right) \]

\[ = \lim_{x \to 1} \left( \frac{4xe^{4x} + e^{4x} \cdot 4 + 4e^{4x} + 0}{2} \right) \]

\[ = \lim_{x \to 1} \left( \frac{4xe^{4x} + e^{4x} \cdot 4 + 4e^{4x} + 0}{2} \right) \]

\[ = \lim_{x \to 1} \left( \frac{4xe^{4x} + e^{4x} \cdot 4 + 4e^{4x} + 0}{2} \right) \]

\[ = \lim_{x \to 1} \left( \frac{4xe^{4x} + e^{4x} \cdot 4 + 4e^{4x} + 0}{2} \right) \]
\[
\frac{4e^4 \cdot 4 + e^4 \cdot 4 + 4e^4}{2} = 12e^4
\]

**Direct Substitution**

**Ex. 3** (Fall 2017)

Calculate the limit or show it DNE.

\[
\lim_{x \to 0} \frac{(1 - \sin(4x))^{6/x}}{x}
\]

**Solution:**

D.S. of \( x = 0 \) gives \( 1^{\pm \infty} \), which is an indeterminate exponent.

Put \( L = \lim_{x \to 0} (1 - \sin(4x))^{6/x} \).

\[
\ln(L) = \lim_{x \to 0} \ln \left[ (1 - \sin(4x))^{6/x} \right]
= \lim_{x \to 0} \left[ \frac{6}{x} \ln(1 - \sin(4x)) \right]
= \lim_{x \to 0} \left[ \frac{6 \ln(1 - \sin(4x))}{x} \right]
\]

\( \frac{0}{0} \)
\[
H = \lim_{x \to 0} \left[ 6 \cdot \frac{1}{1 - \sin(4x)} \cdot (-\cos(4x)) \cdot 4 \right] \\
= \lim_{x \to 0} \left[ -24 \frac{\cos(4x)}{1 - \sin(4x)} \right] = -24
\]

So we have \( \ln(L) = -24 \), whence \( L = e^{-24} \).

**Ex. 4 (Fall 2018)**

Find the largest area of a rectangle whose base is on x-axis and upper vertices are on graph of \( y = e^{-x^2/12} \).

\[y = e^{-x^2/12}\]

**Solution:**
Our objective function (the quantity we want to maximize) is

\[ A(x, y) = 2xy \]

Since \((x, y)\) is on the graph, our constraint equation is

\[ y = e^{-x^2/12} \]

Substituting constraint into objective...

Goal: Find absolute maximum value of

\[ f(x) = 2x e^{-x^2/12} \]

on the interval \([0, \infty)\)

Find the critical #1's:
\[ f'(x) = 2e^{-x^2/12} \left( -\frac{2x}{12} \right) + e^{-x^2/12} \cdot 2 \]

\[ f'(x) = 2e^{-x^2/12} \left( 1 - \frac{x^2}{6} \right) \]

\[ 0 = 2e^{-x^2/12} \left( 1 - \frac{x^2}{6} \right) \]

\[ 0 = 1 - \frac{x^2}{6} \]

\[ x = -\sqrt{6} \quad \text{or} \quad x = \sqrt{6} \quad \checkmark \]

Not in \([0, \infty)\)

So now find the absolute max of \(f\).

We will use the first derivative test.

So now find the absolute max of \(f\).

We will use the first derivative test.

\[ f'(x) = \frac{2e^{-x^2/12}\left(1 - \frac{x^2}{6}\right)}{x} > 0 \]
\[ f'(1) = \text{+} \quad \text{+} = \text{+} \]
\[ f'(3) = \text{+} \quad \text{+} \quad \text{-} = \text{-} \]

Since \( f \) is increasing on \([0, \sqrt{6}]\) and decreasing on \([\sqrt{6}, \infty)\), the absolute max of \( f \) occurs at \( x = \sqrt{6} \). So the max area is \( f(\sqrt{6}) = (2xe^{-x^2/12})\bigg|_{x=\sqrt{6}} = 2\sqrt{6}e^{-1/2} \).

**Ex. 5** (Spring 2018)

The daily output is

\[ Q(L) = 1500L^{2/3} \]

where \( L \) is the size of the labor force, measured in worker-hours. Currently 1000 worker-hours of labor are used each day. Use a linear approximation to estimate the effect on daily output if the labor force is cut to 975 worker-hours.

Solution:
We first find the tangent line to $y = Q(L)$ at $L = 1000$.

$Q(1000) = 1500 \cdot 1000^{2/3} = 1500 \cdot 100 = 150,000$

$Q'(L) = 1500 \cdot \frac{2}{3} L^{-1/3} = \frac{1000}{L^{1/3}}$

$Q'(1000) = \frac{1000}{1000^{1/3}} = \frac{1000}{10} = 100$

So the linearization of $Q(L)$ at $L = 1000$ is

$$g(L) = 150,000 + 100(L - 1000)$$

*Note:* If $L$ is close to 1000, then

$$Q(L) \approx 150,000 + 100(L - 1000)$$

We want to find $\Delta Q$.

$$\Delta Q = Q(975) - Q(1000)$$

new output  old output

$\approx 150,000 + 100(975 - 1000) - 150,000$
= 100 \times (-25) = -2500

So the output decreases by about 2500 units.
Section 5.1: Antiderivatives

Def: We say $F$ is an antiderivative of $f$ on $(a, b)$ if $F'(x) = f(x)$ for all $x$ in $(a, b)$.

Ex:

Suppose $f(x) = \sin(x)$ on $(-\infty, \infty)$. Then what is an antiderivative of $f$?

$F_1(x) = -\cos(x)$

$F_2(x) = -\cos(x) + 5$

$F_3(x) = -\cos(x) + C$

$F_4(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!}$

Thm: Suppose $F$ and $G$ are both antiderivatives of $f$ on $(a, b)$. Then there is some constant $C$ such that
\[ G(x) = F(x) + C \]
for all \( x \) in \( (a, b) \).

**Special Notation**

\[ \int f(x) \, dx \]
means the most general antiderivative of \( f(x) \) wrt. \( x \). (The interval \( (a, b) \) is understood.)

(Your textbook calls this an indefinite integral, but is properly called an antiderivative.)

---

**Ex. 1**

Find each antiderivative:

**Solution:**

(a) \[ \int x \, dx = \frac{1}{2} x^2 + C \]

(b) \[ \int x^2 \, dx = \frac{1}{3} x^3 + C \]
(c) \( \int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \)

(d) \( \int x^{-1/3} \, dx = \frac{3}{2} x^{2/3} + C \)

\[ = \frac{x^{2/3}}{2/3} + C \]

(e) \( \int x^{-1} \, dx = \frac{x^0}{0} + C \) very wrong!

\[ = \ln(x) + C \] almost

---

**Thm:** (Power Rule)

\[ \int x^n \, dx = \begin{cases} 
\frac{x^{n+1}}{n+1} + C & \text{if} \quad n \neq -1 \\
\ln(|x|) + C & \text{if} \quad n = -1 
\end{cases} \]

Does \( \ln(|x|) \) work on \((-\infty, 0) \cup (0, \infty)\)?

**Def:** \( x > 0 \)

\( \ln(|x|) = \ln(x) \)

\[ \frac{d}{dx} \ln(|x|) = \frac{d}{dx} \ln(x) = \frac{1}{x} \]
\[ x < 0: \ln |x| = \ln (-x) \]
\[
\frac{d}{dx} \ln |x| = \frac{d}{dx} \ln (-x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x} \quad \checkmark
\]

\[ y = \ln |x| \]
\[ y = \frac{1}{x} \]

---

**Ex. 2**

Find the following antiderivatives.

**Solution:**

(a) \[ \int \frac{1}{x^3} \, dx = \]

\[ \text{Note: } \int \frac{1}{x^3} \, dx \neq \frac{x}{x^4/4} + C \]

\[ = \int x^{-3} \, dx = \frac{x^{-2}}{-2} + C \]

(b) \[ \int (3x^4 - 5x^{2/3} - x^{-1}) \, dx = \]
Note:
\[
\begin{align*}
\frac{d}{dx} (f + g) &= \frac{df}{dx} + \frac{dg}{dx} \\
\frac{d}{dx} (af) &= a \frac{df}{dx} \\
\int (f + g) \, dx &= \int f \, dx + \int g \, dx \\
\int (af) \, dx &= a \int f \, dx
\end{align*}
\]

\[
\begin{align*}
\int (3x^4) \, dx &= \int (3x^4) \, dx + \int (-5x^{2/3}) \, dx + \int (-x^{-1}) \, dx \\
&= 3 \int x^4 \, dx - 5 \int x^{2/3} \, dx - \int x^{-1} \, dx \\
&= 3 \cdot \frac{x^5}{5} - 5 \cdot \frac{x^{5/3}}{5/3} - \ln |x| + C \\
&= \frac{3}{5} x^5 - 3x^{5/3} - \ln |x| + C
\end{align*}
\]

\[(C) \int \frac{x^3 + \sqrt{2x} + x}{x^3} \, dx = \]
\[ \text{Note: } \int \frac{f(x)}{g(x)} \, dx \neq \frac{\int f(x)}{\int g(x)} \]

\[ = \int \left( 1 + \sqrt{2} x^{-5/2} + x^{-2} \right) \, dx \]

\[ = x + \sqrt{2} \frac{x^{-3/2}}{-3/2} + \frac{x^{-1}}{-1} + C \]

(d) \[ \int (x^2 - 4)^2 \, dx = \]

\[ \text{Note: } \int (x^2 - 4)^2 \, dx \neq \frac{(x^2 - 4)^3}{3} + C \]

\[ \int (x^2 - 4)^2 \, dx \neq \frac{(x^2 - 4)^3}{3 \cdot 2x} + C \]

\[ = \int (x^4 - 8x^2 + 16) \, dx \]

\[ = \frac{x^5}{5} - 8 \cdot \frac{x^3}{3} + 16x + C \]
(c) \[ \int \sec(\theta) (\sec(\theta) + \tan(\theta)) \, d\theta = \]  
\[ = \int (\sec^2(\theta) + \sec(\theta)\tan(\theta)) \, d\theta \]  
\[ = \tan(\theta) + \sec(\theta) + C \]  
(f) \[ \int (e^w + 2\cos(w) - 3\sin(w)) \, dw = \]  
\[ = e^w + 2\sin(w) + 3\cos(w) + C \]  

**Initial Value Problems (IVPs)**

Goal: find an unknown function \( f(x) \) given two pieces of information:
- \( f'(x) \) \( \iff \) gives \( f(x) + C \)
- \( f(a) = b \) \( \iff \) gives value of \( C \)
Position: \[ \frac{d}{dt} s(t) \]

Velocity: \[ \frac{d}{dt} v(t) \]

Acceleration: \[ \frac{d}{dt} a(t) \]

**Ex. 3**

A particle moves along the x-axis with velocity \( v(t) = 9t^2 - 4t \). If the particle is at \( x = 4 \) when \( t = 2 \), find the position of the particle at time \( t = 3 \).

**Solution:**

Recall: \( v(t) = \frac{dx}{dt} = x'(t) \)

This is an IVP and we are given:

- \[ \frac{dx}{dt} = 9t^2 - 4t \] (1)
- \[ x(2) = 4 \] (2)

Our goal is to find \( x(3) \), but we
first need \( x(t) \). Antidifferentiate (i):
\[
x(t) = \int x'(t) \, dt = \int (9t^2 - 4t) \, dt
\]
\[
= 3t^3 - 2t^2 + C
\]
Now use (2) to find value of \( C \).
\[
4 = x(2) = (3t^3 - 2t^2 + C) \bigg|_{t=2}
\]
\[
4 = 24 - 8 + C \implies C = -12
\]
So our position function is:
\[
x(t) = 3t^3 - 2t^2 - 12
\]
So the position at \( t = 3 \) is:
\[
x(3) = 3 \cdot 27 - 2 \cdot 9 - 12 = 51
\]

Ex. 4
Suppose the marginal revenue is
\[
R'(x) = -3x^2 + 4x + 84
\]
Assume \( R(0) = 0 \).
(a) Find the demand function \( p(x) \).
(b) What is the market price if revenue is at a maximum?

**Solution:**

(a) **Recall:** \( R(x) = x \cdot p(x) \)

First find \( R(x) \) by solving the IVP:

\[
\begin{align*}
\text{• } R'(x) &= -3x^2 + 4x + 84 \quad (1) \\
\text{• } R(0) &= 0 \quad (2)
\end{align*}
\]

Antidifferentiate (1), then use (2) to get the "constant of integration".

\[
\begin{align*}
R(x) &= \int R'(x) \, dx = \int (-3x^2 + 4x + 84) \, dx \\
&= -x^3 + 2x^2 + 84x + \hat{C}
\end{align*}
\]

0 = R(0) = \((-x^3 + 2x^2 + 84x + \hat{C})|_{x=0} = \hat{C} \\
\Rightarrow \hat{C} = 0
So the revenue and demand functions are given by:

\[ R(x) = -x^3 + 2x^2 + 84x \]
\[ p(x) = \frac{R(x)}{x} = -x^2 + 2x + 84 \]

(b) Revenue is at a maximum at \(x\) if \(R'(x) = 0\).

\[ R'(x) = -3x^2 + 4x + 84 = 0 \]
\[ -(x-6)(3x+14) = 0 \]
\[ x = 6 \quad \text{or} \quad x = -14/3 \]

So revenue is maximized when \(x = 6\). The market price is

\[ p(6) = (-x^2 + 2x + 84) \bigg|_{x=6} = 60 \]

Ex. 5

The marginal cost of a product is
\[ C'(x) = 6x^2 - 2x + 5 \]
(in hundreds of \$'s). If it costs \$800 to produce 1 unit, what is the cost of producing 5 units?

**Solution:**

We first solve the IVP:

- \[ C'(x) = 6x^2 - 2x + 5 \]  \hspace{1cm} (1)
- \[ C(1) = 8 \]  \hspace{1cm} (2)

Antidifferentiate (1) and use (2) to get the constant of integration.

\[
C(x) = \int C'(x) \, dx = \int (6x^2 - 2x + 5) \, dx
\]
\[
= 2x^3 - x^2 + 5x + \hat{C}
\]

\[
8 = C(1) = (2x^3 - x^2 + 5x + \hat{C})\bigg|_{x=1} = 6 + \hat{C}
\]

\[
\Rightarrow \hat{C} = 2
\]

So our cost function is
\[ C(x) = 2x^3 - x^2 + 5x + 2 \]
So the cost of producing 5 units is

\[ C(5) = 2 \cdot 125 - 25 + 25 + 2 = 252 \]
(Cost is $25,200.)
We want to approximate the area under the graph of \( y = f(x) \) and above the interval \([a, b]\) on the x-axis (We will assume \( f(x) \geq 0 \) and \( f \) is continuous.)

- We will use rectangles whose bases lie on the x-axis to estimate the area.

- First divide \([a, b]\) into \( N \) equal-length subintervals. These subintervals are the bases of the \( N \) rectangles.

- Each rectangle has width
\[ \Delta x = \frac{b-a}{N} \leftarrow \text{total length of } [a,b] \]

\[ \text{N} \leftarrow \# \text{of rectangles} \]

- Determine the endpoints of each of the \( N \) subintervals.

\[ x_0 = a \]

\[ x_1 = a + \Delta x \]

\[ x_2 = a + 2\Delta x \quad (x_2 = x_1 + \Delta x) \]

\[ x_3 = a + 3\Delta x \quad (x_3 = x_2 + \Delta x) \]

\[ \vdots \]

\[ x_N = a + N\Delta x = b \]

- We will choose the height of each rectangle to be the function value at the right endpoint of the corresponding subinterval.

- Total area of rectangles estimates the area under the graph.

(As \# of rectangles increases, the approximation gets better.)
This is called a Riemann sum using right endpoints with $N$ rectangles.

**Methods for determining heights**

Subinterval: $[x_{k-1}, x_k]$

Right endpoints: $h_k = f(x_k)$

Left endpoints: $h_k = f(x_{k-1})$

Midpoint: $h_k = f\left(\frac{x_{k-1} + x_k}{2}\right)$

Midpoint of $[x_{k-1}, x_k]$

* On HW, you have to do all three
* On final exam, you have to do only right endpoints

**Ex. 1**

Let $f(x) = x^3$. Approximate the area under the graph of $y = f(x)$ and above the interval $[0, 2]$ on the $x$-axis.
using right endpoint values and ....

(a) 4 rectangles  (b) 6 rectangles

Solution:

(a) \( N = 4 \)

\[ y = x^3 \]

\[ \Delta x = \frac{2 - 0}{4} = \frac{1}{2} \]

<table>
<thead>
<tr>
<th>Rectangle #</th>
<th>Width</th>
<th>right endpoint</th>
<th>height</th>
<th>area</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{16} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{2}{2} )</td>
<td>( \frac{8}{8} )</td>
<td>( \frac{8}{16} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{27}{8} )</td>
<td>( \frac{27}{16} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{4}{2} )</td>
<td>( \frac{64}{8} )</td>
<td>( \frac{64}{16} )</td>
</tr>
</tbody>
</table>

The approximate area under the graph is the total area of the rectangles.
\[ R_4 = \frac{1 + 8 + 27 + 64}{16} = \frac{100}{16} = \frac{25}{4} \]

right endpoint sum, 4 rectangles

\[ (b) \ N = 6 \]

\[ y = x^3 \]

\[ \Delta x = \frac{2 - 0}{6} = \frac{1}{3} \]

<table>
<thead>
<tr>
<th>Rectangle #</th>
<th>width</th>
<th>right endpoint</th>
<th>height</th>
<th>area $\text{area} = \text{width} \times \text{height}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{27}$</td>
<td>$\frac{1}{81}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{9}{27}$</td>
<td>$\frac{8}{81}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{3}{3}$</td>
<td>$\frac{27}{27}$</td>
<td>$\frac{27}{81}$</td>
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<tr>
<td>4</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{4}{3}$</td>
<td>$\frac{64}{27}$</td>
<td>$\frac{64}{81}$</td>
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<tr>
<td>5</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{5}{3}$</td>
<td>$\frac{125}{27}$</td>
<td>$\frac{125}{81}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{6}{3}$</td>
<td>$\frac{216}{27}$</td>
<td>$\frac{216}{81}$</td>
</tr>
</tbody>
</table>
\[ R_6 = \frac{1 + 9 + 27 + 64 + 125 + 216}{81} = \frac{441}{81} \]

**Special Notation for Sums**

\[ \sum_{k=m}^{n} c_k = c_m + c_{m+1} + c_{m+2} + \ldots + c_n \]

- \(c_k\): terms of sum
- \(m\): lower index
- \(k\): index
- \(n\): upper index

**Ex.**

\[ \sum_{k=3}^{6} c_k = c_3 + c_4 + c_5 + c_6 \]

\[ \sum_{k=3}^{6} k^2 = 3^2 + 4^2 + 5^2 + 6^2 \]

**Limit of Riemann Sums**

Let \(R_n\) be the Riemann sum using right endpoint values and \(N\) rectangles, which is used to approximate the area.
under the graph of \( y = f(x) \) and above the interval \([a, b]\) on the \( x\)-axis. Then we can write \( R_n \) as

\[
R_n = \sum_{k=1}^{N} f(x_k) \Delta x
\]

(\text{where } \Delta x = \frac{b-a}{N} \text{ and } x_k = a + k \Delta x. )

We define the exact area under the graph of \( y = f(x) \) and above the interval \([a, b]\) on the \( x\)-axis as

\[
A = \lim_{N \to \infty} R_n = \lim_{N \to \infty} \sum_{k=1}^{N} f(x_k) \Delta x
\]

(If \( f \) is continuous, this limit exists.)

The number \( A \) is called the integral of \( f \) on \([a, b]\) and it has the special symbol:

\[
A = \int_{a}^{b} f(x) \, dx
\]
"Integral of \( f(x) \) from \( x=a \) to \( x=b \)"

Your text calls this a **definite integral**.

---

**Properties of Integrals**

- \( \int_a^b f(x) \, dx = 0 \) no matter what \( f \) is.

- **Interpretation of integral if \( f(x) \) takes on negative values.**
  \[
  \int_a^b f(x) \, dx = (\text{area above } x\text{-axis}) - (\text{area below } x\text{-axis})
  \]
• **Linearity:**
\[ \int_{a}^{b} (f(x) + g(x)) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \]
\[ \int_{a}^{b} c \cdot f(x) \, dx = c \int_{a}^{b} f(x) \, dx \]

• **Additivity:**
\[ \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx = \int_{a}^{c} f(x) \, dx \]

area from \( a \) to \( b \)
area from \( b \) to \( c \)
area from \( a \) to \( c \)

---

**Ex. 2**

The graph of \( f(x) \) is below.

(Graph consists of line segments)
and semi circles. ) Calculate:

(a) \( \int_0^3 f(x) \, dx \)  
(b) \( \int_0^7 f(x) \, dx \)
(c) \( \int_7^{13} f(x) \, dx \)
(d) \( \int_7^{14} f(x) \, dx \)
(e) \( \int_3^{13} |f(x)| \, dx \)

Solution: \( A_{\text{disk}} = \pi r^2 \)

(c) \( \int_7^{13} \frac{9 \pi}{2} \, dx = \frac{9 \pi}{2} \)

\( \frac{9 \pi}{2} \) - 4 = \( \frac{1}{2} \)

\( \frac{9 \pi}{2} \) - 4 = \( \frac{1}{2} \)

area above \( x \)-axis  
area below \( x \)-axis
(d) \[ \int_{7}^{14} f(x) \, dx = \frac{9\pi}{2} - \frac{\pi}{4} = \frac{17\pi}{4} \]

area above \quad area below
\hspace{1cm} x-axis \quad x-axis

(e) \[ \int_{3}^{13} |f(x)| \, dx = ?? \]

Note: \[ \int_{a}^{b} f(x) \, dx \neq \int_{a}^{b} |f(x)| \, dx \]

Why? \[ \sum |f(x_k)| \Delta x \neq \sum f(x_k) \Delta x \]
\[ |a| + |b| \neq |a + b| \]

We calculate the integral by interpreting as the area under the graph of \( y = |f(x)| \).

Q: If you are given the graph of \( y = f(x) \), how do you get the graph of \( y = |f(x)| \)?
\[
\int_{3}^{13} |f(x)| \, dx = 4 + \frac{9\pi}{2}
\]

Note: \[
\int_{3}^{13} f(x) \, dx = -4 + \frac{9\pi}{2}
\]

So to calculate \[
\int_{a}^{b} |f(x)| \, dx,
\]
simply treat all areas as positive, even those lying below the x-axis.
Section 5.4: Fundamental Theorem of Calculus

**Theorem #1: Fundamental Theorem (Part 1)**
Suppose \( f \) is continuous on \([a, b]\).
If \( F \) is an antiderivative of \( f \) on \((a, b)\),
\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

Recall Notation:
- \( \int_a^b f(x) \, dx \):
  - Integral of \( f \)
  - Area under graph
  - A number
- \( \int f(x) \, dx \):
  - The most general antiderivative of \( f \)
  - Family of functions
So FTC1 tells us that if we have an antiderivative of $f$, then we can use it to find integrals of $f$.

**Q:** Does a general function have an antiderivative?

**A:** No. (But continuous functions do.)

**Q:** If the antiderivative exists, how do you find a useful formula for it?

**A:** Very difficult. (Calculus II.)

**Theorem #2: Fundamental Theorem (part 2)**

Suppose $f$ is continuous on $[a, b]$. Let $x \in [a, b]$ and define

$$A(x) = \int_a^x f(t) \, dt$$

Then $A$ is an antiderivative of $f$ on $(a, b)$. That is, $A'(x) = f(x)$, or
\[ \frac{d}{dx} \left( \int_{a}^{x} f(t) \, dt \right) = f(x) \]

**Note:** Not all continuous functions are differentiable (e.g., \( |x| \), \((x-3)^{\frac{2}{3}}\), etc.). But all cont. functions have antiderivatives.

*Special notation for FTC:

\[ g(x) \bigg|_{a}^{b} = g(b) - g(a) \]

**Ex. 1**

Calculate the following integrals.

(a) \[ \int_{1}^{3} x^3 \, dx \]

\[ \int_{1}^{3} x^3 \, dx = \frac{x^4}{4} \bigg|_{1}^{3} = \frac{3^4}{4} - \frac{1^4}{4} = 20 \]

antiderivative \[ g(3) - g(1) \]
(b) \( \int_{-\pi/4}^{\pi/4} \sec^2(\theta) \, d\theta \)

\[
\int_{-\pi/4}^{\pi/4} \sec(\theta)^2 \, d\theta = \tan(\theta) \bigg|_{-\pi/4}^{\pi/4} = \tan\left(\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right) = 1 - (-1) = 2
\]

(c) \( \int_{0}^{\pi} \sin(\theta) \, d\theta \)

\[
\int_{0}^{\pi} \sin(\theta) \, d\theta = -\cos(\theta) \bigg|_{0}^{\pi} = [-\cos(\pi)] - [-\cos(0)] = 2
\]

(d) \( \int_{0}^{2\pi} \sin(\theta) \, d\theta \)

positive area of 2

negative area of -2
\[ \int_0^{2\pi} \sin(\theta)\,d\theta = -\cos(\theta) \bigg|_0^{2\pi} = (-\cos 2\pi) - (-\cos 0) = (-1) - (-1) = 0 \]

(e) \[ \int_1^2 \frac{1}{x}\,dx \]

\[ \int_1^2 \frac{1}{x}\,dx = \ln |x| \bigg|_1^2 = \ln(2) - \ln(1) = \ln(2) \]

(f) \[ \int_0^2 \frac{1}{x+1}\,dx \]

What is an antiderivative of the function \( f(x) = \frac{1}{x+1} \)?

\[ \frac{d}{dx} \ln(x+1) = \frac{1}{x+1} \cdot 1 = \frac{1}{x+1} \]
Since coefficient of $x$ is 1, chain rule does not make finding the antiderivative any more difficult.

\[
\int_0^2 \frac{1}{x+1} \, dx = \ln(x+1) \bigg|_0^2 = \ln(3) - \ln(1) = \ln(3)
\]

(g) \[\int_0^2 \frac{1}{2x+1} \, dx\]

What is an antiderivative of the function $f(x) = \frac{1}{2x+1}$? Not quite

\[
\frac{d}{dx} \left( \ln(2x+1) \right) = \frac{1}{2x+1} \cdot 2
\]

Our antiderivative is off by a constant factor. So how do we fix this? Divide our candidate by that factor.
\[
\frac{d}{dx} \left( \frac{1}{2} \ln (2x+1) \right) = \frac{1}{2} \cdot \frac{1}{2x+1} \cdot 2 = \frac{1}{2x+1}
\]

\[
\int_{0}^{2} \frac{1}{2x+1} \, dx = \frac{1}{2} \ln (2x+1) \bigg|_{0}^{2} = \frac{1}{2} \ln (5) - \frac{1}{2} \ln (1) = \frac{1}{2} \ln (5)
\]

\[\text{(h) } \int_{0}^{1} (3x+5)^{18} \, dx \]

Looks like \( u^{18} \)

\[
\frac{d}{dx} \left( \frac{(3x+5)^{19}}{19} \right) = \frac{19 \cdot (3x+5)^{18}}{19} \quad \times
\]

\[
\frac{d}{dx} \left( \frac{(3x+5)^{19}}{57} \right) = \frac{19 \cdot (3x+5)^{18} \cdot 3}{57} \quad \checkmark
\]

\[
\int_{0}^{1} (3x+5)^{18} \, dx = \frac{(3x+5)^{19}}{57} \bigg|_{0}^{1} = \frac{8^{19}}{57} - \frac{5^{19}}{57}
\]

\text{Note: This method only works if coefficient of } x \text{ is constant.}
\[ \int \cos(x^2) \, dx \neq \sin(x^2) \cdot \frac{1}{2x} + C \]

\[ \int \cos(x^2) \, dx \neq \sin(x^2) \cdot \frac{1}{x} + C \]

(i) \[ \int_{8/27}^{1} \frac{4t^{4/3} - 10t^{1/3}}{t^2} \, dt \approx \text{algebra} \]

\[ = \int_{8/27}^{1} (4t^{-2/3} - 10t^{-5/3}) \, dt \]

\[ = \left[ 4 \cdot \frac{t^{1/3}}{1/3} - 10 \cdot \frac{t^{-2/3}}{-2/3} \right]_{8/27}^{1} \]

\[ = \left[ 12t^{1/3} + 15t^{-2/3} \right]_{8/27}^{1} \]

Note: \( \left( \frac{8}{27} \right)^{1/3} = \frac{2}{3} \), \( \left( \frac{8}{27} \right)^{-2/3} = \frac{9}{4} \)

\[ = \left[ 12 \cdot 1 + 15 \cdot 1 \right] - \left[ 12 \cdot \frac{2}{3} + 15 \cdot \frac{9}{4} \right] = -\frac{59}{4} \]

Ex. 2
Suppose \( x \geq -1 \). Let
\[
G(x) = \int_{-1}^{x} \sqrt{t^3+1} \, dt
\]

(a) Calculate \( G(1) \).

(b) Calculate \( G'(2) \).

**Solution:**

(a) \( G(1) = \int_{1}^{1} \sqrt{t^3+1} \, dt = 0 \)

(b) Recall FTC2:

Upper limit is \( x \).

Replace dummy variable with \( x \).

\[
\frac{d}{dx} \left( \int_{a}^{x} f(t) \, dt \right) = f(x)
\]

Lower limit is a constant.
So we have

\[ G'(x) = \frac{d}{dx} \left( \int_1^x \sqrt{t^3 + 1} \, dt \right) = \sqrt{x^3 + 1} \]

So we have \( G'(2) = \sqrt{8 + 1} = 3 \).

Ex. 3

Let \( f(x) = \begin{cases} 
\ e^x & \text{if } x < 0 \\
1 - x^2 & \text{if } x \geq 0 
\end{cases} \)

(a) Can we use FTC1 to calculate the integral \( \int_{-1}^{1} f(x) \, dx \)?

(b) Calculate \( \int_{-1}^{1} f(x) \, dx \).

Solution:

(a) Equivalently, we want to determine whether \( f(x) \) is continuous on \([-1, 1]\). Since each piece is continuous on \((-\infty, \infty)\), we only need to check whether \( f \) is
continuous at $x = 0$. We have:

$$\lim_{{x \to 0^-}} f(x) = \lim_{{x \to 0^-}} (e^x) = e^0 = 1$$

$$\lim_{{x \to 0^+}} f(x) = \lim_{{x \to 0^+}} (1-x^2) = 1-0 = 1$$

$$f(0) = (1-x^2)\bigg|_{{x=0}} = 1$$

Since $\lim_{{x \to 0^-}} f(x) = \lim_{{x \to 0^+}} f(x) = f(0)$, $f$ is continuous at $x = 0$.

(b)

\[ y = e^x \quad y = 1 - x^2 \quad y = f(x) \]

\[ \int_{-1}^{1} f(x) \, dx = \int_{-1}^{0} f(x) \, dx + \int_{0}^{1} f(x) \, dx \]

\[ \int_{-1}^{0} f(x) \, dx = \int_{-1}^{0} e^x \, dx = e^x \bigg|_{-1}^{0} \]
\[ e^0 - e^{-1} = 1 - \frac{1}{e} \]

\[ \int_0^1 f(x) \, dx = \int_0^1 (1 - x^2) \, dx = \left( x - \frac{x^3}{3} \right) \bigg|_0^1 \]

\[ = (1 - \frac{1}{3}) - (0 - 0) = \frac{2}{3} \]

So the total area is

\[ \int_{-1}^1 f(x) \, dx = 1 - \frac{1}{e} + \frac{2}{3} = \frac{5}{3} - \frac{1}{e} \]
Section 5.5: Substitution Rule

\[ \int 2x e^{x^2} \, dx \quad \text{vs.} \quad \int e^{x^2} \, dx \]

The derivative of the "inside" function \( x^2 \) is sitting outside as a multiplicative factor.

* This should remind you of chain rule!

\[
\int 2x e^{x^2} \, dx = e^{x^2} + C
\]

\[
\frac{d}{dx} (e^{x^2}) = e^{x^2} \cdot 2x
\]

\[
\int 2x \cos(x^2) \, dx = \sin(x^2) + C
\]

\[
\frac{d}{dx} (\sin(x^2)) = \cos(x^2) \cdot 2x
\]

\underline{Thm: (Substitution Rule)}

Suppose \( F' = f \). Then...
\[ \int f(u(x)) u'(x) \, dx = F(u(x)) + C \]

*In practice, this rule is used by treating \( \frac{du}{dx} \) as if it were a fraction.*

**Example:**

Calculate \( \int 2x e^{x^2} \, dx \).

**Solution:**

**Rule of thumb:** The function \( u \) should be chosen to be an “inside” function whose derivative is outside as a multiplicative factor.

Let \( u = x^2 \). Now we construct a “translation table” from \( x \) to \( u \).

\[
\begin{align*}
  u &= x^2 \\
  \frac{du}{dx} &= 2x \\
  dx &= \frac{du}{2x}
\end{align*}
\]

- Calculate derivative of \( u \).
- Solve “algebraically” for \( dx \).
Now use translation table to rewrite integral in terms of $u$ only:

\[
\int 2x e^{x^2} \, dx = \int 2x e^u \cdot \frac{du}{2x} = \int e^u \, du
\]

This is much easier than the original.

Now find an antiderivative.

\[
\int e^u \, du = e^u + C = e^{x^2} + C
\]

translate back to $x$.

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**Ex. 1**

Calculate the following antiderivatives.

**Solution:**

(a) \[\int 3x^2 \sin (x^3) \, dx\]

Let $u = x^3$. Translation table:
Now translate integral to $u$.

$$\int 3x^2 \sin(x^3) \, dx = \int 3x^2 \sin(u) \cdot \frac{du}{3x^2}$$

$$= \int \sin(u) \, du = -\cos(u) + C = -\cos\left(x^3\right) + C$$

(b) $\int x \left(x^2 - 7\right)^{-1/2} \, dx$

New rule of thumb: Choose $u$ so that any constant multiple of its derivative is outside as a multiplicative factor.

Let $u = x^2 - 7$. Translation table:

$$\begin{align*}
u &= x^2 - 7 \\
\frac{du}{dx} &= 2x \\
\frac{dx}{du} &= \frac{1}{2x}
\end{align*}$$
Now translate integral to $u$.

$$\int x (x^2-7)^{-1/2} \, dx = \int x u^{-1/2} \cdot \frac{du}{2x} = \int \frac{1}{2} u^{-1/2} \, du$$

$$= \frac{1}{2} \cdot \frac{u^{1/2}}{1/2} + C = u^{1/2} + C = (x^2-7)^{1/2} + C$$

(c) \[ \int \frac{9x^2 - 6}{x^3 - 2x + 5} \, dx \]

Let \( u = x^3 - 2x + 5 \). Translation table:

\[
\begin{align*}
  u &= x^3 - 2x + 5 \\
  \frac{du}{dx} &= 3x^2 - 2 \\
  dx &= \frac{du}{3x^2 - 2}
\end{align*}
\]

Now translate integral to $u$.

$$\int \frac{9x^2 - 6}{x^3 - 2x + 5} \, dx = \int \frac{9x^2 - 6}{u} \cdot \frac{du}{3x^2 - 2}$$

$$= \int \frac{3(3x^2 - 2)}{u} \cdot \frac{du}{3x^2 - 2} = \int \frac{3}{u} \, du$$
\[ = 3 \ln |u| + C = 3 \ln |x^3 - 2x + 5| + C \]

Note: If the integrand is changed even slightly, substitution may not work.

\[
\int \frac{9x^2 - 7}{x^3 - 2x + 5} \, dx = \int \frac{9x^2 - 7}{u} \cdot \frac{du}{3x^2 - 2}
\]

\[ = \int 3 \left( 3x^2 - \frac{7}{3} \right) \cdot \frac{du}{u} \cdot \frac{du}{3x^2 - 2} \]

Too bad!

Ex. 2

Calculate \[ \int \tan(\theta) \, d\theta. \]

Solution:

Rewrite in terms of sine and cosine.

\[ \int \tan(\theta) \, d\theta = \int \frac{\sin(\theta)}{\cos(\theta)} \, d\theta \]

**Method 1** let \[ u = \cos(\theta). \]
Now translate integral to $u$.

$$
\int \frac{\sin(\theta)}{\cos(\theta)} \, d\theta = \int \frac{\sin(\theta)}{u} \cdot \frac{du}{-\sin(\theta)} = -\int \frac{1}{u} \, du
$$

$$
= -\ln |u| + C = -\ln |\cos(\theta)| + C
$$

**Method 2** Let $u = \sin(\theta)$.

Now translate the integral to $u$.

$$
\int \frac{\sin(\theta)}{\cos(\theta)} \, d\theta = \int \frac{u}{\cos(\theta)} \cdot \frac{du}{\cos(\theta)} = \int \frac{u}{\cos^2(\theta)} \, du
$$
Note: The variable $\theta$ is still present. We must express $\cos(\theta)^2$ in terms of $u$ before we may proceed. (In this case, we write $\cos(\theta)^2 = 1 - \sin(\theta)^2 = 1 - u^2$.)

\[
\int \frac{u}{\cos(\theta)^2} \, du = \int \frac{u}{1-u^2} \, du
\]

Integral is in terms of $u$. Good! Where do we go from here?

We need a second substitution. Let $w = 1 - u^2$. Translation table:

\[
\begin{align*}
w &= 1 - u^2 \\
dw &= -2u \\
du &= \frac{dw}{-2u}
\end{align*}
\]

Now to translate integral to $w$.

\[
\int \frac{u}{1-u^2} \, du = \int \frac{u}{w} \cdot \frac{dw}{-2u} = -\frac{1}{2} \int \frac{1}{w} \, dw
\]
\[ = -\frac{1}{2} \ln |u| + C = -\frac{1}{2} \ln |1-u^2| + C \]
\[ = -\frac{1}{2} \ln |1-\sin(\theta)^2| + C \]
\[ = -\frac{1}{2} \ln |\cos(\theta)^2| + C = -\ln |\cos(\theta)| + C \]

**Substitution Rule for Integrals**

We need to change three components.

1. **Integrand**
2. **Differential**
3. **Limits of integration**

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**Ex. 3**

Calculate \( \int_{0}^{2} x^3 (x^4+1)^{\frac{1}{3}} \, dx \).

**Solution:**

Let \( u = x^4 + 1 \). Translation table:

\[ u = x^4 + 1 \]  \[ \text{New limits of integration:} \]
Translate integral to $u$:

$$
\int_0^2 x^3 (x^4 + 1)^{1/3} \, dx = \int_1^u \frac{1}{x^3} \cdot u^{1/3} \cdot \frac{du}{4x^3} \cdot \frac{1}{4} \cdot \frac{u^{4/3}}{4/3} \bigg|_1^{17} 
$$

Note: Do not re-express in terms of the old variable $x$!

"No more $x$, only $u" \quad \text{❤️}

$$
= \left[ \frac{3}{16} u^{4/3} \right]_1^{17} = \left( \frac{3}{16} \cdot 17^{4/3} \right) - \left( \frac{3}{16} \cdot 1^{4/3} \right)
$$

$$
= \frac{3}{16} \left( 17^{4/3} - 1 \right)
$$
Ex. 4

Calculate the area under the graph of \( f(x) = \frac{(\ln x)^2}{x} \) from \( x = 1 \) to \( x = e \).

Solution:
We need to calculate the integral
\[
\int_{1}^{e} \frac{(\ln x)^2}{x} \, dx
\]
Let \( u = \ln(x) \). Translation table:

<table>
<thead>
<tr>
<th>( u = \ln(x) )</th>
<th>New limits of integration</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{du}{dx} = \frac{1}{x} )</td>
<td>( x = 1 \Rightarrow u = \ln(1) = 0 )</td>
</tr>
<tr>
<td>( dx = x , du )</td>
<td>( x = e \Rightarrow u = \ln(e) = 1 )</td>
</tr>
</tbody>
</table>

Now translate integral to \( u \).
\[
\int_{1}^{e} \frac{(\ln x)^2}{x} \, dx = \int_{0}^{1} \frac{u^2}{x} \cdot x \, du
\]
\[
= \int_{0}^{1} u^2 \, du = \frac{1}{3} u^3 \bigg|_{0}^{1} = \frac{1}{3} - 0 = \frac{1}{3}
\]
Ex. 5: Calculate \( \int_{0}^{\pi/4} \tan(\theta)^3 \sec(\theta)^2 \, d\theta \)

**Solution:**

Let \( u = \tan(\theta) \). Then we have...

<table>
<thead>
<tr>
<th>( u = \tan(\theta) )</th>
<th>( \theta = 0 ) ( \Rightarrow ) ( u = \tan(0) ) ( \Rightarrow u = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{du}{d\theta} = \sec(\theta)^2 )</td>
<td>( \theta = \frac{\pi}{4} ) ( \Rightarrow ) ( u = \tan\left(\frac{\pi}{4}\right) ) ( \Rightarrow u = 1 )</td>
</tr>
<tr>
<td>( d\theta = \frac{du}{\sec(\theta)^2} )</td>
<td></td>
</tr>
</tbody>
</table>

Now transform the integral.

\[
\int_{0}^{\pi/4} \tan(\theta)^3 \sec(\theta)^2 \, d\theta = \int_{0}^{1} u^3 \, du = \left. \frac{1}{4} u^4 \right|_{0}^{1} = \frac{1}{4} - 0 = \frac{1}{4}
\]

**Note:** We could have also used \( u = \sec(\theta) \).....
\[ \frac{du}{d\theta} = \sec(\theta) \tan(\theta) \]

\[ d\theta = \frac{du}{\sec(\theta) \tan(\theta)} \]

\[ \theta = \frac{\pi}{4} \Rightarrow u = \sec \left( \frac{\pi}{4} \right) \]

\[ u = \sqrt{2} \]

Now transform the integral:

\[ \int_{0}^{\pi/4} \tan(\theta)^3 \sec(\theta)^2 \, d\theta = \int_{1}^{\sqrt{2}} \tan(\theta)^3 \, u^2 \cdot \frac{du}{\sec(\theta) \tan(\theta)} \]

\[ = \int_{1}^{\sqrt{2}} \tan(\theta)^2 \, u \, du = \int_{1}^{\sqrt{2}} (\sec(\theta)^2 - 1) \, u \, du \]

\[ \text{trigonometric identity} \]

\[ = \int_{1}^{\sqrt{2}} (u^2 - 1) \, u \, du = \int_{1}^{\sqrt{2}} (u^3 - u) \, du \]

\[ = \left( \frac{u^4}{4} - \frac{u^2}{2} \right) \bigg|_{1}^{\sqrt{2}} = \left( \frac{4}{4} - \frac{2}{2} \right) - \left( \frac{1}{4} - \frac{1}{2} \right) \]

\[ = (0) - \left( -\frac{1}{4} \right) = \frac{1}{4} \]