1. For each part, write “TRUE” or “FALSE”. You are not required to show work, but you may use the provided space for scratch work. For each part, there is no partial credit.

(a) \( \ln(3) - \ln(11) = \frac{\ln(3)}{\ln(11)} \)

(b) The domain of \( f(x) = \sqrt[3]{x - 4} \) is all real numbers.

(c) The lines \( 9x + y = 1 \) and \( x - 9y = 4 \) are perpendicular to each other.

(d) The equations \( 2 \ln(x) = 0 \) and \( \ln(x^2) = 0 \) have the same solutions.

(e) \( \cos \left( \frac{5\pi}{6} \right) = -\frac{\sqrt{3}}{2} \)

Solution
(a) False. The correct identity is \( \ln(3) - \ln(11) = \ln \left( \frac{3}{11} \right) \).
(b) True. Every real number has an odd root.
(c) True. The slope of the line \( 9x + y = 1 \) is \( m_1 = -9 \) and the slope of the line is \( x - 9y = 4 \) is \( m_2 = \frac{1}{9} \). Since \( m_1 m_2 = -1 \), the lines are perpendicular.
(d) False. The equation \( 2 \ln(x) = 0 \) has solution \( x = 1 \). The equation \( \ln(x^2) = 0 \) has solutions \( x = 1 \) and \( x = -1 \). (The identity \( \ln(x^b) = b \ln(x) \) is true only if \( x > 0 \).)
(e) True. The reference angle for \( \frac{5\pi}{6} \) is \( \frac{\pi}{6} \), and \( \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2} \). Since the angle \( \frac{5\pi}{6} \) lies in the second quadrant, its cosine is negative.

2. For each part, write your answer on the provided line. You are not required to show work, but you may use the provided space for scratch work. For each part, there is no partial credit.

(a) Calculate \( \lim_{x \to 5} \left( \frac{x - 5}{x^2 - 2x - 15} \right) \) or determine that the limit does not exist.

(b) Calculate the derivative of \( f(x) = e^x \sin(x) \). Do not simplify your answer.

(c) Find the slope of the line tangent to the graph of \( y = 3 \ln(x) - 6\sqrt{x} \) at \( x = 3 \).

(d) The number \( N \) of bacteria at time \( t \) grows exponentially, so that \( N(t) = N_0 e^{kt} \). Suppose an initial population of 100 bacteria grows to 500 after 2 hours. How many hours does it take for an initial population of 150 bacteria to grow to 300?

Your answer should be written exactly in terms of logarithms.

(e) Find the value of \( k \) that makes \( f(x) \) continuous at \( x = 1 \). If no such value of \( k \) exists, write “does not exist”.

\[ f(x) = \begin{cases} 
  k \cos(\pi x) - 3x^2 & \text{if } x \leq 1 \\
  8e^x - k \ln(x) & \text{if } x > 1 
\end{cases} \]

(f) Calculate \( \lim_{x \to 0} \left( \frac{\sin(9x)}{\sin(16x)} \right) \) or determine that the limit does not exist.

(g) Suppose \( f(4) = -8 \) and \( f'(4) = 3 \). Let \( g(x) = f \left( \frac{1}{4} x^2 \right) \). Find \( g'(4) \). If it is impossible to find \( g'(4) \) with the given information, write “not enough information”.

(h) The graph of \( y = f(x) \) is given below. Find all values of \( a \) in the interval \((-4, 4)\) for which \( \lim_{x \to a} f(x) \) does not exist. If there are no such values of \( a \), write “does not exist”.

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Solution

(a) Cancel common factors.

$$\lim_{x \to 5} \left( \frac{x - 5}{x^2 - 2x - 15} \right) = \lim_{x \to 5} \left( \frac{x - 5}{(x - 5)(x + 3)} \right) = \lim_{x \to 5} \left( \frac{1}{x + 3} \right) = \frac{1}{8}$$

(b) Use product rule.

$$f'(x) = e^x \sin(x) + e^x \cos(x)$$

(c) Observe that

$$\frac{dy}{dx} = 3 \cdot \frac{1}{x} - 6 \cdot \frac{1}{2} x^{-1/2} = \frac{3}{x} - \frac{3}{\sqrt{x}}$$

Hence the slope of the tangent line is

$$\left. \frac{dy}{dx} \right|_{x=3} = \frac{3}{3} - \frac{3}{\sqrt{3}} = 1 - \sqrt{3}$$

(d) We are given that if $N_0 = 100$ then $N(2) = 500$. Hence 500 = 100$e^{2k}$, and solving for $k$ gives $k = \frac{1}{2} \ln(5)$. Now we want to solve the equation $300 = 150e^{kt}$ for $t$ with the known value of $k$. This gives $t = \frac{2 \ln(2)}{\ln(5)}$.

(e) We require that the left-limit, right-limit, and function value at $x = 1$ be equal to ensure continuity at $x = 1$.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (k \cos(\pi x) - 3x^2) = k \cos(\pi) - 3 = -k - 3$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (8e^x - k \ln(x)) = 8e^1 - k \ln(1) = 8e$$

$$f(1) = (k \cos(\pi x) - 3x^2) \bigg|_{x=1} = k \cos(\pi) - 3 = -k - 3$$

Hence we must have $-k - 3 = 8e$, or $k = -8e - 3$. 
(f) Use the special limit \( \lim_{\theta \to 0} \left( \frac{\sin(\theta)}{\theta} \right) = 1 \) and some algebra.

\[
\lim_{x \to 0} \left( \frac{\sin(9x)}{\sin(16x)} \right) = \lim_{x \to 0} \left( \frac{\sin(9x) \cdot 16x}{9x \cdot \sin(16x)} \cdot \frac{9}{16} \right) = 1 \cdot 1 \cdot \frac{9}{16} = \frac{9}{16}
\]

(g) Use chain rule.

\[g'(x) = f' \left( \frac{1}{4}x^2 \right) \cdot \frac{1}{2}x \]

Hence \( g'(4) = f'(4) \cdot 2 = 6 \).

(h) The values of \( a \) for which \( \lim_{x \to a} f(x) \) does not exist are \( a = -1 \) and \( a = 1 \) only. (At both of these values of \( a \), the left-limit and right-limit are not equal.)

10 pts

3. Find an equation of the line tangent to the curve

\[
\frac{5x}{y} = 4x + y^3
\]

at the point \((1,1)\). **Any form of the equation of a line is acceptable.**

**Solution**

Differentiate each side of the equation with respect to \( x \) using implicit differentiation.

\[
\frac{5 \cdot y - 5x \cdot y'}{y^2} = 4 + 3y^2 \cdot y'
\]

Substituting the point \((x, y) = (1, 1)\) gives \( 5 - 5y' = 4 + 3y' \), whence \( y' = \frac{1}{8} \). Hence the tangent line has equation

\[y - 1 = \frac{1}{8}(x - 1)\]

10 pts

4. Let \( g(x) = 6 - \frac{9}{x} \). Calculate \( g'(3) \) directly from the limit definition of the derivative.

*If you simply quote a rule, you will receive no credit. You must use the limit definition and show all work.*

**Solution**

Start with the definition of derivative and compute the limit using algebra.

\[
g'(3) = \lim_{h \to 0} \left( \frac{g(3 + h) - g(3)}{h} \right) = \lim_{h \to 0} \left( \frac{(6 - \frac{9}{3+h}) - (6 - \frac{9}{3})}{h} \right) = \lim_{h \to 0} \left( \frac{3 - \frac{9}{3+h}}{h} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{3(3+h) - 9}{h(3+h)} \right) = \lim_{h \to 0} \left( \frac{3h}{h(3+h)} \right) = \lim_{h \to 0} \left( \frac{3}{3+h} \right) = \frac{3}{3+0} = 1
\]
5. Show that the following equation is satisfied for at least one value of $x$ in the interval $(0, 1)$.

$$\sqrt{12x^4 + 9x^2 + 4} = 4x^3 + 2x$$

You must identify any theorem you use by name and clearly justify its use.

Solution

Equivalently, we show that the equation

$$\sqrt{12x^4 + 9x^2 + 4} - (4x^3 + 2x) = 0$$

has a solution in the interval $(0, 1)$. To that end, let $f(x) = \sqrt{12x^4 + 9x^2 + 4} - (4x^3 + 2x)$. Observe that $f$ is continuous on $[0, 1]$, $f(0) = 2$, and $f(1) = -1$. Since 0 is a value between $f(0)$ and $f(1)$, the Intermediate Value Theorem (IVT) guarantees that $f(x) = 0$ has a solution in $(0, 1)$.

6. Consider the function $f(x)$ below.

$$f(x) = \begin{cases} 
\frac{4 - \sqrt{2x + 10}}{x - 3}, & x \neq 3 \\
1, & x = 3
\end{cases}$$

Is $f(x)$ continuous at $x = 3$? Explain your answer. You must use proper calculus to give a complete and clear justification for your answer.

Solution

First we calculate the limit of $f(x)$ as $x \to 3$.

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \left( \frac{4 - \sqrt{2x + 10}}{x - 3} \right) = \lim_{x \to 3} \left( \frac{4 - \sqrt{2x + 10}}{x - 3} \cdot \frac{4 + \sqrt{2x + 10}}{4 + \sqrt{2x + 10}} \right)$$

$$= \lim_{x \to 3} \left( \frac{16 - (2x + 10)}{(x - 3)(4 + \sqrt{2x + 10})} \right) = \lim_{x \to 3} \left( \frac{-2(x - 3)}{(x - 3)(4 + \sqrt{2x + 10})} \right)$$

$$= \lim_{x \to 3} \left( \frac{-2}{4 + \sqrt{2x + 10}} \right) = \frac{-2}{4 + \sqrt{2 \cdot 3 + 10}} = \frac{-1}{4}$$

Observe that $\lim_{x \to 3} f(x) = f(3) = 1$, and so $f$ is not continuous at $x = 3$.

7. Calculate the derivative of $f(x) = \frac{\ln(e^{4x} + 6)}{9\tan(x) - \pi^9}$. Do not simplify your answer.

Solution

Start with quotient rule. To differentiate the numerator, use chain rule twice.

$$f'(x) = \frac{\frac{1}{e^{4x} + 6} \cdot e^{4x} \cdot 4}{(9\tan(x) - \pi^9)^2} \cdot (9\tan(x) - \pi^9) - \ln(e^{4x} + 6) \cdot 9\sec(x)^2$$