1. At a certain factory, the daily output is

\[ Q(L) = 1500L^{2/3} \]

units, where \( L \) denotes the size of the labor force measured in worker-hours. Currently 1,000 worker-hours of labor are used each day. Use a linear approximation to estimate the effect on the daily output if the labor force is cut to 975 worker-hours.

**Solution**

The linear approximation formula is

\[ \Delta Q = Q(L_0 + \Delta L) - Q(L_0) \approx Q'(L_0)\Delta L \]

For this problem, \( L_0 = 1000 \) and \( \Delta L = 975 - 1000 = -25 \). We also have

\[ Q'(L) = 1000L^{-1/3} \implies Q'(L_0) = 100 \]

Hence we have

\[ \Delta Q \approx (100)(-25) = -2500 \]

So the output decreases by approximately 2500 units.

2. Find an equation of the line tangent to the curve

\[ x^3 + e^{xy} = 3y + 9 \]

at the point \((2, 0)\).

**Solution**

Implicitly differentiating the equation with respect to \( x \) gives

\[ 3x^2 + e^{xy}(xy' + y) = 3y' \]

Substituting \( x = 2 \) and \( y = 0 \) gives

\[ 12 + 1 \cdot (2y' + 0) = 3y' \implies y' = 12 \]

Hence the equation of the tangent line is

\[ y - 0 = 12(x - 2) \]

3. On the axes provided, draw the graph of a function \( f(x) \) with domain \([-8, 8]\) that satisfies all of the following properties.

- \( f(-2) = -3 \) and \( f'(-6) = 0 \)
- asymptotes: \( x = -2 \) and \( y = -3 \)
- \( f \) is decreasing on: \((-8, -2), (-2, 2)\)
- \( f \) is increasing on: \((2, 8)\)
• $f$ is concave down on: $(-6, -2), (6, 8)$
• $f$ is concave up on: $(-8, -6), (-2, 6)$

Solution
There are many such solutions. Here is one.

4. The total surface area of a cube is changing at a rate of 12 in$^2$/s when the length of one of the sides is 10 in. At what rate is the volume of the cube changing at that time?

You must include correct units as part of your answer.

Solution
Let $x$ be the side length of the cube. Then the total surface area and volume of the cube are

$$S = 6x^2, \quad V = x^3$$

Differentiating with respect to time $t$ gives

$$\frac{dS}{dt} = 12x \frac{dx}{dt}, \quad \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

These four equations hold for all time. Now we substitute the information relevant to the specific time, i.e., $x = 10$ and $\frac{dS}{dt} = 12$.

$$S = 600, \quad V = 1000, \quad 12 = 120 \frac{dx}{dt}, \quad \frac{dV}{dt} = 300 \frac{dx}{dt}$$
Solving for $\frac{dx}{dt}$ in the third equation gives $\frac{dx}{dt} = \frac{12}{120} = \frac{1}{10}$. Substituting into the fourth equation gives

$$\frac{dV}{dt} = 300 \cdot \frac{1}{10} = 30$$

Hence the volume of the cube is increasing at a rate of $30 \text{ in}^3/\text{sec}$.

**5.** Find the minimum and maximum values of

$$f(x) = 2x^3 - 3x^2 - 12x + 18$$

on the interval $[-3, 3]$.

*Note that $f$ may also be factored as $f(x) = (x^2 - 6)(2x - 3)$.*

**Solution**

Since $f$ is differentiable everywhere (it is a polynomial), the only critical numbers are solutions to $f'(x) = 0$.

$$0 = f'(x) = 6x^2 - 6x - 12 = 6(x - 2)(x + 1) \implies x = -1, 2$$

Now we find the values of $f$ at the critical numbers and endpoints $x = -3, 3$. (We may use the factored form of $f$ to make the arithmetic easier.)

$$f(x) = (x^2 - 6)(2x - 3)$$

$$f(-3) = (3)(-9) = -27$$

$$f(-1) = (-5)(-5) = 25$$

$$f(2) = (-2)(1) = -2$$

$$f(3) = (3)(3) = 9$$

Hence the absolute minimum is $-27$ and the absolute maximum is $25$.

**6.** Calculate each of the following limits or show it does not exist. Show all work.

**6 pts**

(a) $\lim_{x \to 1} \left( \frac{x^{1/4} - 1}{e^{2x} - e^2} \right)$

(b) $\lim_{x \to 1} \left( (x - 1) \tan \left( \frac{\pi x}{2} \right) \right)$

**Solution**

(a) Standard application of L’Hospital’s Rule.

$$\lim_{x \to 1} \left( \frac{x^{1/4} - 1}{e^{2x} - e^2} \right) = \lim_{x \to 1} \left( \frac{\frac{1}{4}e^{-3/4}x^{-3/4}}{2e^{2x}} \right) = \frac{\frac{1}{4} \cdot e^2}{2e^2} = \frac{1}{8e^2}$$
(b) We write the product as a quotient and then use L'Hospital’s Rule.

\[
\lim_{{x \to 1}} \left( (x - 1) \tan \left( \frac{\pi x}{2} \right) \right) = \lim_{{x \to 1}} \left( \frac{(x - 1) \sin \left( \frac{\pi x}{2} \right)}{\cos \left( \frac{\pi x}{2} \right)} \right)
\]

\[
= \lim_{{x \to 1}} \left( \frac{(x - 1) \left( \frac{\pi}{2} \right) + \sin \left( \frac{\pi x}{2} \right)}{-\sin \left( \frac{\pi x}{2} \right)} \right)
\]

\[
= 0 \cdot \frac{\pi}{2} + 1 = -\frac{2}{\pi}
\]

16 pts 7. You are constructing a rectangular box with a total surface area (six sides) of 450 in\(^2\). The length of the box is three times its width. Find the dimensions of the box, measured in inches, with the largest possible volume.

**You must give a full justification for your answer using methods taught in this course. You must also demonstrate that your answer really does give the maximum volume.**

**Solution**

Let \(\ell, w,\) and \(h\) denote the length, width, and height of the box, respectively. We want to maximize the volume

\[ V(\ell, w, h) = \ell wh \]

Since \(V\) is a function of 3 variables, we must eliminate 2 of the variables. We will solve for all variables in terms of the width \(w\). We immediately have that \(\ell = 3w\). The total surface area is given by

\[ S = 2\ell w + 2\ell h + 2wh \]

Substituting \(S = 450\) and \(\ell = 3w\) gives

\[ 450 = 6w^2 + 8wh \]

Now we solve for \(h\) in terms of \(w\).

\[ h = \frac{225 - 3w^2}{4w} \]

Rewriting \(\ell\) and \(h\) in terms of \(w\) in our volume function shows that \(V\) may be written as the single variable function

\[ V(w) = 3w \cdot w \cdot \frac{225 - 3w^2}{4w} = 3 \left( \frac{225w - 3w^3}{4} \right) \]

We now maximize \(V(w)\) on the interval \(w \in (0, \sqrt{75})\). (The interval is found by considering the extreme cases \(\ell = 0, w = 0, h = 0\) as degenerate boxes. Neither endpoint may be included since then the surface area would be 0, not 450.)
Since $V(w)$ is differentiable everywhere (it is a polynomial), the only critical numbers are solutions to $V'(w) = 0$.

$$0 = V'(w) = \frac{3}{4} (225 - 9w^2) \implies w^2 = \frac{225}{9} = 25 \implies w = 5$$

(The solution $w = -5$ is not physical since width cannot be negative.) Now observe that $V''(w) = -\frac{3}{4} (18w) < 0$ for all $w > 0$. Hence the graph of $V(w)$ is concave down for $w > 0$, and so $w = 5$ gives a maximum value of $V(w)$.

The optimal dimensions are $\ell = 15$, $w = 5$, and $h = 7.5$ (all measured in inches).

8. Consider the function $f$ and its derivatives below.

$$f(x) = \frac{x^2}{x^2 - 1} \quad f'(x) = \frac{-2x}{(x^2 - 1)^2} \quad f''(x) = \frac{6x^2 + 2}{(x^2 - 1)^3}$$

(a) Find all horizontal asymptotes of $f$.

(b) Find all vertical asymptotes of $f$. Then at each vertical asymptote you find, calculate the corresponding one-sided limits of $f$.

(c) Find where $f$ is decreasing and find where $f$ is increasing. Then calculate all points of local extrema, classifying each as either a local minimum, a local maximum, or neither.

(d) Find where $f$ is concave down and find where $f$ is concave up. Then calculate all points of inflection.

**Solution**

(a) Horizontal asymptotes are found by computing the limit of $f$ as $x \to \pm \infty$.

$$\lim_{x \to \pm \infty} \left( \frac{x^2}{x^2 - 1} \right) = \lim_{x \to \pm \infty} \left( \frac{1}{1 - \frac{1}{x^2}} \right) = \frac{1}{1 - 0} = 1$$

Hence the only horizontal asymptote is the line $y = 1$.

(b) Since $f$ is continuous on its domain, the only candidate vertical asymptotes are the lines $x = -1$ and $x = 1$ (since these are the only $x$-values not in the domain of $f$). Direct substitution of either $x = -1$ or $x = 1$ into $f(x)$ gives the expression $\frac{1}{0}$, which is undefined by indicates that all of the corresponding one-sided limits at both $x = -1$ and $x = 1$ are infinite. Hence $x = -1$ and $x = 1$ are vertical asymptotes. Now we may compute the limits using sign analysis.

$$\lim_{x \to -1^-} \left( \frac{x^2}{x^2 - 1} \right) = \bigoplus \infty = \infty$$

$$\lim_{x \to -1^+} \left( \frac{x^2}{x^2 - 1} \right) = \bigoplus \infty = -\infty$$

$$\lim_{x \to 1^-} \left( \frac{x^2}{x^2 - 1} \right) = \bigoplus \infty = -\infty$$

$$\lim_{x \to 1^+} \left( \frac{x^2}{x^2 - 1} \right) = \bigoplus \infty = \infty$$
(c) Since \( f \) is differentiable on its domain, the only first-order critical numbers are solutions to \( f'(x) = 0 \).

\[
-\frac{2x}{(x^2 - 1)^2} = 0 \implies -2x = 0 \implies x = 0
\]

Recall that since \( x = -1 \) and \( x = 1 \) are not in the domain of \( f \), we must include such \( x \)-values on our sign chart also.

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So \( f \) is decreasing on \((0, 1)\) and \((1, \infty)\); \( f \) is increasing on \((-\infty, -1)\) and \((-1, 0)\). There is no local minimum; there is a local maximum at \((0, 0)\).

(d) Since \( f \) is twice-differentiable on its domain, the only second-order critical numbers are solutions to \( f''(x) = 0 \).

\[
\frac{6x^2 + 2}{(x^2 - 1)^3} = 0 \implies 6x^2 + 2 = 0 \implies \text{no solution}
\]

Recall that since \( x = -1 \) and \( x = 1 \) are not in the domain of \( f \), we must include such \( x \)-values on our sign chart also.

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Hence \( f \) is concave down on \((-1, 1)\); \( f \) is concave up on \((-\infty, -1)\) and \((1, \infty)\). There are no inflection points.