1. The parts of this question are independent of each other.

(a) Given the function \(g(x)\), state the definition of \(g'(4)\).

(b) Let \(F(x) = \frac{1}{3x - 5}\). Calculate \(F'(2)\) directly from the definition. Show all work.

\[
\text{If you simply quote a rule, you will receive no credit. You must use the definition of derivative.}
\]

Solution

(a) \(g'(4) = \lim_{h \to 0} \left( \frac{g(4 + h) - g(4)}{h} \right)\)

(b) Start with the definition of derivative, then simplify and cancel.

\[
F'(2) = \lim_{h \to 0} \left( \frac{F(2 + h) - F(2)}{h} \right) = \lim_{h \to 0} \left( \frac{1}{3(2+h)-5} - \frac{1}{h} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{\frac{1}{3h+1} - \frac{1}{h}}{h} \right) = \lim_{h \to 0} \left( \frac{1 - (3h + 1)}{h(3h + 1)} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{-3}{3h + 1} \right) = \frac{-3}{0+1} = -3
\]

2. For each part, calculate \(f'(x)\).

After calculating the derivative, do not simplify your answer.

(a) \(f(x) = \frac{x^{-1}x^{8/3}}{4\sqrt{x^2}}\)

(b) \(f(x) = (x + \sqrt{5x - 6})^{1/4}\)

(c) \(f(x) = \frac{x^2e^x}{\ln(x) - \cos(x)}\)

Solution

(a) Simplifying the exponents, we observe that \(f(x) = \frac{1}{4}x\). Hence \(f'(x) = \frac{1}{4}\).

(b) Use power rule and chain rule (twice!).

\[
f'(x) = \frac{1}{4} \left( x + \sqrt{5x - 6} \right)^{-3/4} \cdot \left( 1 + \frac{1}{2} (5x - 6)^{-1/2} \cdot 5 \right)
\]

(c) Use quotient rule.

\[
f'(x) = \frac{(\ln(x) - \cos(x)) \cdot (x^2e^x + 2xe^x) - (x^2e^x) \cdot \left( \frac{1}{x} + \sin(x) \right)}{(\ln(x) - \cos(x))^2}
\]
3. For each limit, calculate the value or show that it does not exist. Show all work.

(a) \( \lim_{x \to 0} \left( \frac{(2x + 9)^2 - 81}{x} \right) \)

(b) \( \lim_{x \to 3^-} \left( \frac{|x - 3|}{x - 3} \right) \)

(c) \( \lim_{x \to 1} \left( \frac{5 - \sqrt{32 - 7x}}{x - 1} \right) \)

Solution

(a) We have the following work.

\[
= \lim_{x \to 0} \left( \frac{4x^2 + 36x + 81 - 81}{x} \right) = \lim_{x \to 0} \left( \frac{4x^2 + 36x}{x} \right) = \lim_{x \to 0} (4x + 46) = 36
\]

(b) If \( x \to 3^- \), then we may assume that \( x < 3 \), or \( x - 3 < 0 \). For such values of \( x \), we have that \( |x - 3| = -(x - 3) \). So now we have

\[
\lim_{x \to 3^-} \left( \frac{|x - 3|}{x - 3} \right) = \lim_{x \to 3^-} \left( \frac{-(x - 3)}{x - 3} \right) = -1
\]

(c) We have the following work.

\[
= \lim_{x \to 1} \left( \frac{5 - \sqrt{32 - 7x}}{x - 1} \right) \cdot \frac{5 + \sqrt{32 - 7x}}{5 + \sqrt{32 - 7x}}
\]

\[
= \lim_{x \to 1} \left( \frac{25 - (32 - 7x)}{(x - 1)(5 + \sqrt{32 - 7x})} \right) = \lim_{x \to 1} \left( \frac{7x - 7}{(x - 1)(5 + \sqrt{32 - 7x})} \right)
\]

\[
= \lim_{x \to 1} \left( \frac{7(x - 1)}{(x - 1)(5 + \sqrt{32 - 7x})} \right) = \lim_{x \to 1} \left( \frac{7}{5 + \sqrt{32 - 7x}} \right)
\]

\[
= \frac{7}{5 + \sqrt{32 - 7}} = \frac{7}{10}
\]

4. The graph of a function \( f(x) \) is shown below.
Read each question carefully. You are not required to show work.

(a) State where \( f(x) \) is not continuous in the interval \((-5, 5)\).

(b) State where \( f(x) \) is not differentiable in the interval \((-5, 5)\).

(c) State where \( f'(x) = 0 \) in the interval \((-5, 5)\).

(d) State where \( f'(x) < 0 \) in the interval \((-5, 5)\).

Solution

(a) \( x = -3, \ x = -1 \)

(b) \( x = -3, \ x = -1, \ x = 3 \)

Recall that continuity is necessary for differentiability. So any points of discontinuity are also points of non-differentiability. At \( x = 3 \), the graph exhibits a sharp corner, which means the function is not differentiable there.

(c) all \( x \)-values in the interval \((-3, -1)\) or \( x = 1 \).

Recall that if \( f'(a) = 0 \), then the graph of \( y = f(x) \) has a horizontal tangent line at \( x = a \). That is, the slope of the graph of \( f(x) \) is 0.

(d) on each of the intervals \((-5, -3), (-1, 1), \) and \((3, 5)\)
5. Each part of this question refers to the function $f$ below, where $a$ and $b$ are constants.

$$f(x) = \begin{cases} \frac{\sin(ax)}{x}, & x < 0 \\ 2x + 3, & 0 \leq x < 1 \\ b, & x = 1 \\ \frac{x^2 - 1}{x - 1}, & 1 < x \end{cases}$$

For each of the following parts, you must give a full, clear justification for your answer. You must use proper methods taught in this course.

(a) Find the value of the constant $a$ so that $f$ is continuous at $x = 0$. If this is not possible, explain why.

(b) Find the value of the constant $b$ so that $f$ is continuous at $x = 1$. If this is not possible, explain why.

Solution

(a) We require that the left-limit, right-limit, and function value all be equal at $x = 0$.

We have the following.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left( \frac{\sin(ax)}{x} \right) = \lim_{x \to 0^-} \left( a \cdot \frac{\sin(ax)}{ax} \right) = a \cdot 1 = a$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2x + 3) = 3$$

$$f(0) = (2x + 3)|_{x=0} = 3$$

So we must have that $a = 3$.

(b) We require that the left-limit, right-limit, and function value all be equal at $x = 1$.

We have the following.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x + 3) = 5$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left( \frac{x^2 - 1}{x - 1} \right) = \lim_{x \to 1^+} \left( \frac{(x - 1)(x + 1)}{x - 1} \right) = \lim_{x \to 1^+} (x + 1) = 2$$

$$f(0) = b$$

So we must have that $5 = 2 = b$, which is impossible.

(It is impossible to find such a value of $b$ because $\lim_{x \to 1} f(x)$ does not exist.)

6. Find an equation of each line that is both tangent to the graph of $y = 4x^2 - 3x - 1$ and parallel to the line $y = 13x - 5$. 

6.10 pts
Solution
The slope of the desired tangent line is 13 (parallel lines have equal slope). Hence we must solve the equation $f'(x) = 13$ where $f(x) = 4x^2 - 3x - 1$.

$$f'(x) = 8x - 3 = 13 \implies x = 2$$

Observe that $f(2) = 9$. Hence the desired tangent line has slope 13 and passes through the point $(2, 9)$. An equation of this line is

$$y - 9 = 13(x - 2)$$

7. Suppose $f(x)$ is continuous on the interval $[0, 6]$. Selected values of $f$ are given in the table below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>-3</td>
<td>2</td>
<td>12</td>
<td>-5</td>
<td>-1</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

On what intervals is the graph of $y = f(x)$ guaranteed to intersect the $x$-axis? Explain your answer.

List as many intervals as possible, but no two intervals can overlap.

Solution
Since $f$ is continuous on $[0, 6]$, the intermediate value theorem (IVT) applies to this interval and any subintervals. Hence the IVT guarantees that $f(x) = 0$ on any interval on which the values of $f$ change sign (either from positive to negative or negative to positive). There are three such intervals: $(0, 1)$, $(2, 3)$, and $(4, 5)$.

8. A coffee vendor has collected data on the price of coffee in her store over the last year. The price of coffee $t$ weeks since when the data collection began was

$$p(t) = 0.02t^2 - 0.1t + 6$$

dollars per pound.

For each part, you must give correct units as part of your answer.

(a) How much did the price of one pound of coffee increase in the first ten weeks after the data collection began?

(b) What was the average rate at which the price of one pound of coffee changed over the same ten-week period mentioned in part (a)?

The vendor also found that, in a given week, the local consumers bought approximately

$$D(p) = \frac{2500}{p^2 + 1}$$

pounds of coffee when the price was $p$ dollars per pound. That is, $D$ is the weekly demand of the consumers.

(c) Calculate $D'(7)$ and explain its precise meaning in the given context.

(d) At what rate was the weekly demand for coffee changing with respect to time exactly ten weeks after data collection began?
Solution

(a) The change in price is \( \Delta p = p(10) - p(1) = 7 - 6 = 1 \). The units are dollars.

(b) The average rate is \( \frac{\Delta p}{\Delta t} = \frac{p(10) - p(0)}{10 - 0} = \frac{1}{10} \). The units are dollars per week.

(c) First find the derivative of \( D \) using chain rule or quotient rule.

\[
D'(p) = -\frac{5000p}{(p^2 + 1)^2}
\]

Now substitute \( p = 7 \).

\[
D'(7) = -\frac{5000 \cdot 7}{50^2} = -\frac{5000 \cdot 7}{2500} = -14
\]

The units are pounds per dollar. When the price of one pound of coffee was 7 dollars, the weekly demand was decreasing at a rate of 14 pounds per dollar. In other words, if this rate were constant, this would mean that each additional 1-dollar increase in the price would result in a loss of 14 pounds in the weekly demand.

(d) Observe that \( D = D(p(t)) \). So we must use chain rule.

\[
\frac{dD}{dt} = \frac{d}{dt}D(p(t)) = D'(p(t)) \cdot p'(t)
\]

Now substitute \( t = 10 \).

\[
\left. \frac{dD}{dt} \right|_{t=10} = D'(p(10)) \cdot p'(10)
\]

Now we compute each of these numbers. The following have already been computed in previous parts.

\[
p(10) = 7
\]

\[
D'(p(10)) = D'(7) = -14
\]

The following have not already been computed.

\[
p'(t) = 0.04t - 0.1
\]

\[
p'(10) = 0.04 \cdot 10 - 0.1 = 0.3
\]

Hence we find that

\[
\left. \frac{dD}{dt} \right|_{t=10} = D'(p(10)) \cdot p'(10) = (-14) \cdot (0.3) = -4.2
\]

The units are pounds per week.