1. Calculate $\lim_{x \to 5} \left( \frac{x^2 - 3x - 10}{x^2 - x - 20} \right)$. If the limit does not exist, write “DNE”.

**Solution**
Cancel common factors.

\[
\lim_{x \to 5} \left( \frac{x^2 - 3x - 10}{x^2 - x - 20} \right) = \lim_{x \to 5} \left( \frac{(x - 5)(x + 2)}{(x - 5)(x + 4)} \right) = \lim_{x \to 5} \left( \frac{x + 2}{x + 4} \right) = \frac{5 + 2}{5 + 4} = \frac{7}{9}
\]

2. Calculate $\lim_{x \to 0} \left( \frac{\sin^2(4x)}{x^2} \right)$. If the limit does not exist, write “DNE”.

**Solution**
Use the special limit $\lim_{\theta \to 0} \left( \frac{\sin(a\theta)}{a\theta} \right) = 1$.

\[
\lim_{x \to 0} \left( \frac{\sin^2(4x)}{x^2} \right) = \left( \lim_{x \to 0} \frac{\sin(4x)}{x} \right)^2 = \left( \lim_{x \to 0} \frac{\sin(4x)}{4x} \cdot 4 \right)^2 = (1 \cdot 4)^2 = 16
\]

3. Calculate $\lim_{x \to 4} \left( \frac{3 - \sqrt{2x + 1}}{x - 4} \right)$. If the limit does not exist, write “DNE”.

**Solution**
First rationalize the numerator.

\[
\lim_{x \to 4} \left( \frac{3 - \sqrt{2x + 1}}{x - 4} \right) = \lim_{x \to 4} \left( \frac{9 - (2x + 1)}{(x - 4)(3 + \sqrt{2x + 1})} \right) = \lim_{x \to 4} \left( \frac{-2(x - 4)}{(x - 4)(3 + \sqrt{2x + 1})} \right) = \lim_{x \to 4} \left( \frac{-2}{3 + \sqrt{9}} \right) = \frac{-2}{3 + 3} = \frac{-1}{3}
\]

4. Let $f(x) = \frac{\ln(x)}{10 - x^3}$. Calculate $f'(x)$. Do not simplify your final answer.

**Solution**
Use quotient rule.

\[
f'(x) = \frac{1}{x} \cdot (10 - x^3) - \ln(x) \cdot (-3x^2)
\]

5. Let $f(x) = \sqrt{\cos(3 + x^5)}$. Calculate $f'(x)$. Do not simplify your final answer.

**Solution**
Use chain rule twice.
\[ f'(x) = \frac{1}{2} \left( \cos(3 + x^5) \right)^{-1/2} \cdot (-\sin(3 + x^5)) \cdot 5x^4 \]

6. Solve the inequality \( \frac{3x - 6}{x + 4} > 0 \). Write your answer using interval notation.

**Solution**
We solve the inequality using the method of sign charts. The cut points for our number line are \( x = 2 \) (obtained by solving \( 3x - 6 = 0 \)) and \( x = -4 \) (obtained by solving \( x + 4 = 0 \)). Hence we examine each of the three subintervals: \( (-\infty, -4) \), \( (-4, 2) \), and \( (2, \infty) \). We test the truth of the inequality on each of these subintervals by testing one \( x \)-value in each subinterval. Testing \( x = -5 \), \( x = 0 \), and \( x = 3 \), we find that the inequality is false only for \( x = 0 \). Hence the inequality is true for all \( x \) in the set \( (-\infty, -4) \cup (2, \infty) \).

7. Find an equation of the line tangent to the graph of \( y = \tan(2x) \) at \( x = \frac{\pi}{8} \). Any form of the equation of a line is acceptable.

**Solution**
The tangent line must pass through the point \( \left( \frac{\pi}{8}, \tan\left(\frac{\pi}{4}\right) \right) = \left( \frac{\pi}{8}, 1 \right) \). Now we find the derivative using chain rule.
\[
 f'(x) = \sec(2x)^2 \cdot 2 \n\]
Hence the slope of the tangent line is \( f'(\frac{\pi}{8}) = 2 \sec\left(\frac{\pi}{4}\right)^2 = 4 \). The equation of the tangent line is:
\[
 y - 1 = 4 \left( x - \frac{\pi}{8} \right) 
\]

8. Find all critical numbers of \( f(x) = 2 - (x^2 - 2x)^{1/3} \) in the interval \( (-\infty, \infty) \). If there are no critical numbers, write “NONE”.

**Solution**
Critical numbers come in two types: where \( f'(x) \) does not exist or where \( f'(x) = 0 \). First note that since \( g(x) = x^{2/3} \) is not differentiable at \( x = 0 \), \( f(x) \) is not differentiable when \( x^2 - 2x = 0 \) (i.e., at both \( x = 0 \) and \( x = 2 \)). Now we solve \( f'(x) = 0 \).
\[
 0 = f'(x) = -\frac{2x - 2}{3(x^2 - 2x)^{2/3}} \Rightarrow x = 1 
\]
Hence \( f(x) \) has three critical numbers: \( x = 0 \), \( x = 1 \), and \( x = 2 \).

9. Let \( f(x) = 2x^2 - 5x + 7 \). Use the limit definition of the derivative to calculate \( f'(x) \). If you simply quote a derivative rule without using the limit definition, you will receive no credit.
Solution

Start with the definition of derivative and compute the limit using algebra.

\[ f'(x) = \lim_{h \to 0} \left( \frac{f(x + h) - f(x)}{h} \right) = \lim_{h \to 0} \left( \frac{2(x + h)^2 - 5(x + h) + 7 - (2x^2 - 5x + 7)}{h} \right) \]

\[ = \lim_{h \to 0} \left( \frac{2x^2 + 4xh + 2h^2 - 5x - 5h + 7 - 2x^2 + 5x - 7}{h} \right) = \lim_{h \to 0} \left( \frac{4xh + 2h^2 - 5h}{h} \right) \]

\[ = \lim_{h \to 0} \left( \frac{h(4x + 2h - 5)}{h} \right) = \lim_{h \to 0} (4x + 2h - 5) = 4x - 5 \]

12 pts 10. Find the absolute extreme values of \( f(x) = x + \frac{9}{x} \) on \([1, 18]\).

Solution

We first find the critical numbers of \( f \). Since \( f \) is differentiable on its domain, all critical numbers satisfy \( f'(x) = 0 \).

\[ 0 = f'(x) = 1 - \frac{9}{x^2} \implies x = -3 \text{ or } x = 3 \]

The only critical number in the interval \((1, 18)\) is \( x = 3 \). Now we compare the critical values and the endpoint values: \( f(1) = 10 \), \( f(3) = 6 \), and \( f(18) = 18.5 \). Hence on the interval \([1, 18]\), the absolute minimum value of \( f \) is 6 and the absolute maximum value is 18.5.

12 pts 11. Find all points on the graph of \( y = x \ln(x) \) where the tangent line is horizontal.

Solution

A horizontal line has slope 0 and the slope of the tangent line is given by the derivative. Hence we must solve the equation \( f'(x) = 0 \).

\[ 0 = f'(x) = 1 + \ln(x) \implies x = e^{-1} \]

Hence the point on the graph with a horizontal tangent is \((e^{-1}, e^{-1} \ln(e^{-1})) = (e^{-1}, -e^{-1})\).

12 pts 12. Find the absolute extreme values of \( f(x) = (6 - x)e^x \) on \([0, 6]\). \( Hint: 2 < e < 3 \).

Solution

We first find the critical numbers of \( f \). Since \( f \) is differentiable on its domain, all critical numbers satisfy \( f'(x) = 0 \).

\[ 0 = f'(x) = (5 - x)e^x \implies x = 5 \]

The only critical number in the interval \((0, 6)\) is \( x = 5 \). Now we compare the critical values and the endpoint values: \( f(0) = 6 \), \( f(5) = e^5 \), and \( f(6) = 0 \). Hence on the interval \([0, 6]\),
the absolute minimum value of $f$ is 0 and the absolute maximum value is $e^5$.

13. Find the values of $a$ and $b$ that make $f$ continuous at $x = 1$ or determine that no such values of $a$ and $b$ exist.

$$f(x) = \begin{cases} 
-3x + ax^2, & x < 1 \\
b, & x = 1 \\
4ax - 1, & x > 1
\end{cases}$$

You must show all work and use proper calculus methods and notation to receive full credit. Your explanation must be clear and coherent.

**Solution**

First we calculate the left-limit, right-limit, and function value at $x = 1$.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (-3x + ax^2) = -3 + a$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4ax - 1) = 4a - 1$$
$$f(1) = b$$

To make $f$ continuous at $x = 1$, the left-limit, right-limit, and function value at $x = 1$ must all be equal. Hence we must have

$$-3 + a = 4a - 1 = b$$

Solving for $a$ in $-3 + a = 4a - 1$ gives $a = -2/3$, and then solving for $b$ in $4a - 1 = b$ gives $b = -11/3$. 
