1. Find an equation of the line tangent to the graph of \( y = 2x^2 - 3x + 1 \) at \( x = 1 \).

**Solution**

Differentiating \( y \) gives

\[ y' = 4x - 3 \]

If \( x = 1 \), then \( y = 2(1)^2 - 3(1) + 1 = 0 \) and \( y' = 4(1) - 3 = 1 \). Hence the equation of the tangent line is

\[ y - 0 = 1(x - 1) \]

2. Show that the equation

\[ x^{2/3} = 2x^2 + 2x - 2 \]

has at least one solution in the interval \([0, 1]\). Explain your answer.

**Solution**

Let \( f(x) = 2x^2 + 2x - 2 - x^{2/3} \). Since \( f \) is a sum of power functions, \( f \) is continuous on its domain, which includes the interval \([0, 1]\). Observe that \( f(0) = -2 \) and \( f(1) = 1 \). Since 0 is between \(-2\) and 1, the intermediate value theorem guarantees the existence of a number \( c \) in the interval \([0, 1]\) such that \( f(c) = 0 \). The same number \( c \) satisfies the desired equation.

3. Find all real solutions to the following equation.

\[ 2 \ln(x) = \ln\left(\frac{x^5}{5-x}\right) - \ln\left(\frac{x^3}{2+x}\right) \]

**Solution**

Using the logarithm law \( a \ln(b) = \ln(b^a) \) on the left side of the equation and the logarithm law \( \ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right) \) on the right side of the equation gives

\[ \ln\left(x^2\right) = \ln\left(\frac{x^5}{5-x}\right) \]

Simplifying the right side gives

\[ \ln\left(x^2\right) = \ln\left(\frac{x^2(2+x)}{5-x}\right) \]

Exponentiating each side gives

\[ x^2 = \frac{x^2(2+x)}{5-x} \]

Cross-multiplication and some algebra gives

\[ 0 = x^2(5-x) - x^2(2+x) = x^2(5-x-2-x) = x^2(3-2x) \]
Observe that $x = 0$ cannot be a solution to the original equation since $\ln(0)$ is undefined. Hence $x^2 \neq 0$ and we must have $3 - 2x = 0$. The only solution is $x = \frac{3}{2}$.

4. Find the values of the constants $a$ and $b$ so that the following function is continuous for all $x$. If this is not possible, explain why.

$$f(x) = \begin{cases} 
ax + b, & x < 1 \\
-2, & x = 1 \\
3\sqrt{x} + b, & x > 1 
\end{cases}$$

You must give a full, clear justification for your answer. You must use proper methods taught in this course.

**Solution**

The first two “pieces” of $f(x)$ are continuous for all $x$ regardless of the values of $a$ and $b$ since polynomials are continuous for all $x$. The “piece” $3\sqrt{x} + b$ is continuous regardless of the value of $b$ as long as $x \geq 0$. Hence each piece is continuous on each of its “pieces” separately on the respective intervals. We need only force continuity at $x = 1$ to guarantee $f$ is continuous for all $x$. Hence we must choose $a$ and $b$ such that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\lim_{x \to 1^-} (ax + b) = \lim_{x \to 1^+} (3\sqrt{x} + b) = -2$$

$$a + b = 3 + b = -2$$

Hence $b = -5$ and $a = 3$.

5. On the set of axes provided below, sketch the graph of a function $f(x)$ that satisfies all of the following properties.

- the domain of $f$ is all real numbers
- $\lim_{x \to -4^+} f(x) = f(-4)$ and $f$ is not continuous at $x = -4$
- $\lim_{x \to 1} f(x)$ exists and $f$ is not continuous at $x = 1$
- $f$ is continuous and not differentiable at $x = 5$

**Solution**

There are many possible solutions. This is perhaps the simplest one.
6. For each part, calculate $f'(x)$.

After calculating the derivative, do not simplify your answer.

(a) $f(x) = \frac{7x^3}{3x^{1/2}x^5}$

(b) $f(x) = -\cos(x)\ln(x)$

(c) $f(x) = \frac{\csc(x) + 4x^3}{e^x - e^5}$

**Solution**

(a) Observe that $f(x) = \frac{7}{3}x^{-5/2}$. Hence $f'(x) = \frac{7}{3} \left( -\frac{5}{2} \right) x^{-7/2}$

(b) Use product rule.

\[
f'(x) = - \left( \cos(x) \cdot \frac{1}{x} + (-\sin(x)) \cdot \ln(x) \right)
\]

(c) Use quotient rule.

\[
f'(x) = \frac{(-\csc(x)\cot(x) + 12x^2)(e^x - e^5) - (\csc(x) + 4x^3)(e^x)}{(e^x - e^5)^2}
\]

7. The parts of this question are independent of each other.

(a) Given the function $g(x)$, state the definition of $g'(x)$.

(b) Let $f(x) = \sqrt{6x + 1}$. Calculate $f'(1)$ directly from the definition. Show all work.

If you simply quote a rule, you will receive no credit. You must use the definition of derivative.

**Solution**

(a) $g'(x) = \lim_{h\to0} \frac{g(x + h) - g(x)}{h}$
(b) Start with the definition of derivative, then simplify and cancel.

\[ f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\sqrt{6(1+h) + 7} - \sqrt{7}}{h} \]
\[ = \lim_{h \to 0} \left( \frac{\sqrt{6h + 7} - \sqrt{7}}{h} \cdot \frac{\sqrt{6h + 7} + \sqrt{7}}{\sqrt{6h + 7} + \sqrt{7}} \right) \]
\[ = \lim_{h \to 0} \frac{(6h + 7) - 7}{h(\sqrt{6h + 7} + \sqrt{7})} = \lim_{h \to 0} \frac{6h}{h(\sqrt{6h + 7} + \sqrt{7})} \]
\[ = \lim_{h \to 0} \frac{6}{\sqrt{6h + 7} + \sqrt{7}} = \frac{6}{\sqrt{7} + \sqrt{7}} = \frac{3}{\sqrt{7}} \]

8. For each limit, calculate the value or show that it does not exist. Show all work.

\[ \text{(a) } \lim_{x \to 7} \left( \frac{\frac{1}{7} - \frac{1}{x}}{x - 7} \right) \]
\[ \text{(b) } \lim_{x \to 0} \left( \frac{\sin(7x)}{\tan(2x)} \right) \]
\[ \text{(c) } \lim_{x \to -1} \left( \frac{|x + 1|}{x + 1} \right) \]

**Solution**

(a) We have the following work.

\[ = \lim_{x \to 7} \left( \frac{\frac{1}{7} - \frac{1}{x}}{x - 7} \cdot \frac{7x}{7x} \right) = \lim_{x \to 7} \left( \frac{x - 7}{7x(x - 7)} \right) = \lim_{x \to 7} \left( \frac{1}{7x} \right) = \frac{1}{49} \]

(b) We have the following work.

\[ = \lim_{x \to 0} \left( \frac{\sin(7x)}{7x} \cdot \frac{2x}{\sin(2x)} \cdot \cos(2x) \cdot \frac{7x}{2x} \right) \]
\[ = \left( \lim_{x \to 0} \frac{\sin(7x)}{7x} \right) \cdot \left( \lim_{x \to 0} \frac{2x}{\sin(2x)} \right) \cdot \left( \lim_{x \to 0} \cos(2x) \cdot \frac{7}{2} \right) \]
\[ = 1 \cdot 1 \cdot 1 \cdot \frac{7}{2} = \frac{7}{2} \]

(c) We have the following work.

\[ \lim_{x \to -1} \left( \frac{|x + 1|}{x + 1} \right) = \lim_{x \to -1} \left( \frac{-(x + 1)}{x + 1} \right) = -1 \]
\[ \lim_{x \to -1^+} \left( \frac{|x + 1|}{x + 1} \right) = \lim_{x \to -1^+} \left( \frac{+(x + 1)}{x + 1} \right) = +1 \]

The one-sided limits are not equal, thus the desired limit does not exist.