How do stochastic Epidemic Models Differ from Deterministic Epidemic Modes?

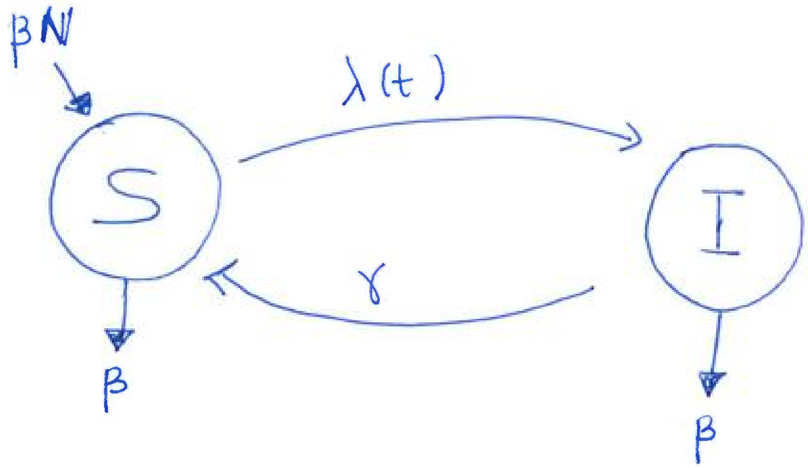
Deterministic

- Formulated in terms of a system of ODEs or DEs.
- Its solution ~~s~~ is a function of time or space and is generally uniquely dependent on the initial data.

Stochastic

- Formulated as a stochastic process with a collection of random variables.
- Its solution is a probability distribution for each of the random variables.
- Often used to show the variability inherent due to the demographics or environment variability are particularly important when quantities in the processes are small (small population size or initial number of infectives).

Deterministic Vs stochastic Models (Discrete time)



Assumptions:

- No deaths due to disease
- No vertical transmission of the disease (all new borns are in S)
- Population size, $N(t)$, remains constant; ($N(t) = S(t) + I(t) \equiv N$ is constant)

Where:

$\lambda(I) = \frac{\alpha}{N} I(t)$ is the force of infection.

~~$\lambda(t)$~~

(# contacts that result in infection per susceptible individual per unit time)

$\alpha \Delta t$ = # successful contacts made by one infectious individual during the time interval Δt

$\beta \Delta t$ = # births or deaths per individual during the time interval Δt

$\gamma \Delta t$ = (Removal number) # individuals that recover in the time interval Δt and become susceptible again.

Δt = fixed time interval

DE system:

$$+ \overbrace{\beta N \Delta t}^{\beta(S+I)\Delta t} - \beta S \Delta t + \gamma I \Delta t$$

$$(1) \quad \begin{cases} S(t+\Delta t) = S(t)(1 - \lambda(I)\Delta t) + \overbrace{(\beta \Delta t + \gamma \Delta t) I(t)} \\ I(t+\Delta t) = I(t)(1 - \beta \Delta t - \gamma \Delta t) + \lambda(I)\Delta t\, S(t) \end{cases}$$

with

$t = n\Delta t$ for $n = 0, 1, 2, \ldots$

$S(0) > 0$, $I(0) > 0$ and $S(0) + I(0) \equiv N$ constant.

Properties of the Model:

Steady states: (Substitute value of $\lambda(I)$): ~~and~~

S.S. are the solns. of:

$$(1) \quad S_\infty = S_\infty\left(1 - \frac{\alpha}{N} I_\infty \Delta t\right) + (\beta+\gamma) I_\infty \Delta t$$

$$(2) \quad I_\infty = I_\infty\left(1 - \beta\Delta t - \gamma\Delta t\right) + \frac{\alpha}{N} I_\infty S_\infty \Delta t$$

$$(3) \quad I_\infty + S_\infty = N$$

Solving system (1) – (3) we have that,

$$\left(S_\infty^{(1)}, I_\infty^{(1)}\right) = (N, 0) \leftarrow \text{Disease free Equilibrium}$$

$$\left(S_\infty^{(2)}, I_\infty^{(2)}\right) = \left(\frac{\beta+\gamma}{\alpha} N, \left[1 - \frac{\beta+\gamma}{\alpha}\right] N\right) \leftarrow \text{Endemic Equilibrium (Positive Equilibrium)}$$

(Note that $\left(S_\infty^{(2)}, I_\infty^{(2)}\right)$ exists only if $R_0 = \dfrac{\alpha}{\beta+\gamma} > 1$)

Thm: (i) If $R_0 \le 1$, then solutions to (1) approach the disease-free equilibrium: $\lim_{t\to\infty} I(t) = 0$, $\lim_{t\to\infty} S(t) = N$.

(ii) If $R_0 > 1$, solutions to (1) approach the endemic equilibrium.

$S + I = N \quad \Rightarrow \quad S = N - I$ (Recall Δt is fixed)

$$\Rightarrow$$

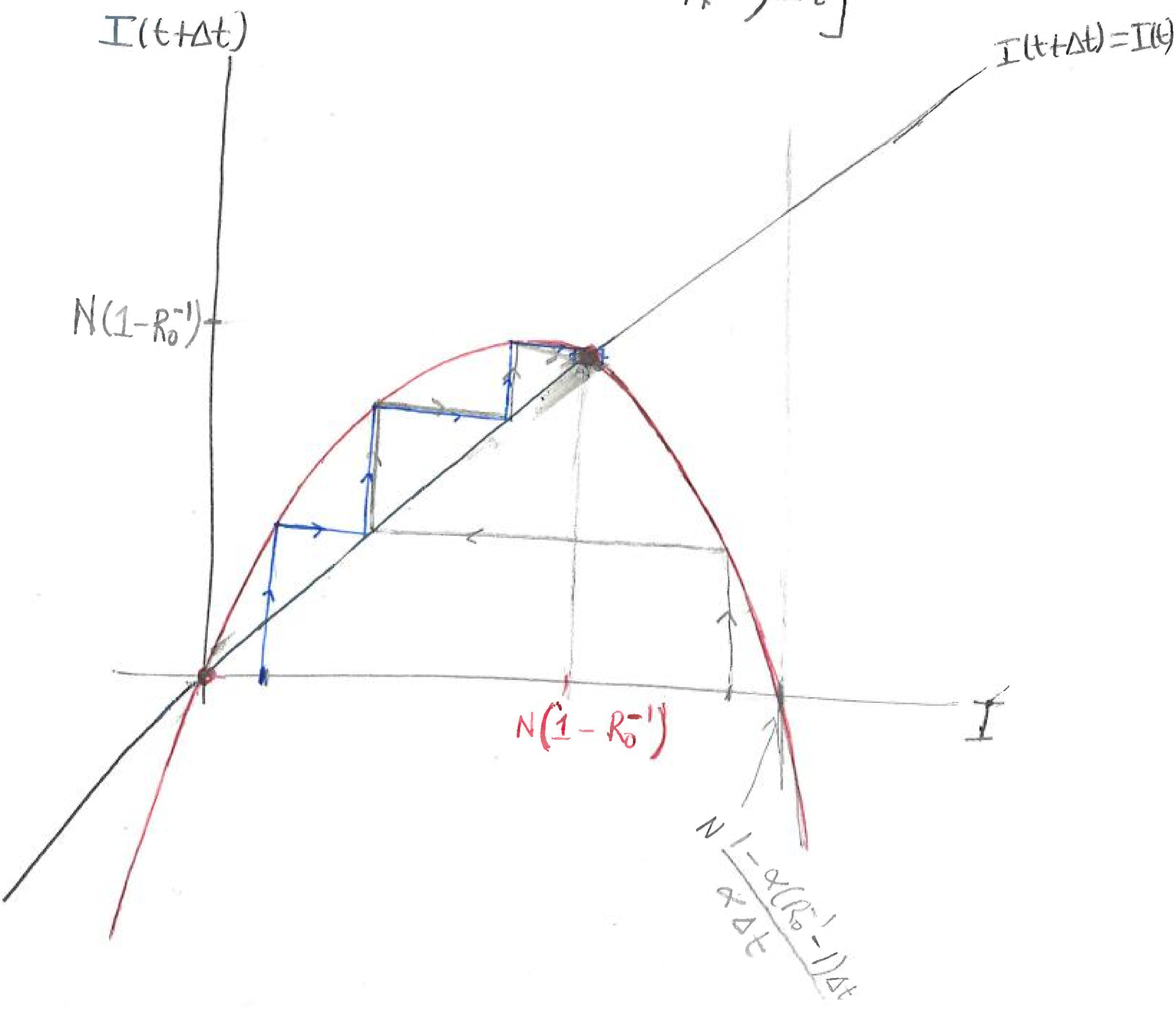
$$I(t+\Delta t) = I\left(1 - (\beta+\gamma-\alpha)\Delta t\right) - \frac{\alpha}{N} I^2 \Delta t$$

$$= I\left[1 - (\beta+\gamma-\alpha)\Delta t - \frac{\alpha}{N} I \Delta t\right]$$

$$= I\left[1 - \alpha\left(R_0^{-1} - 1\right)\Delta t - \frac{\alpha}{N} I \Delta t\right]$$

When $R_0 > 1$

$$= I\left[1 - \left(\alpha\left(R_0^{-1} - 1\right) + \frac{\alpha}{N} I\right)\Delta t\right]$$



~~Stochastic SIR Model~~

Markov Chain (Discrete Markov Chain)

A stochastic process $X = \{X(t) : t \in \{0, \Delta t, 2\Delta t, \dots\}\}$ is called a discrete Markov chain if

$$P(X(t+\Delta t) = i+1 \mid X(0) = 0, \dots, X(t) = i)$$

$$= P(X(t+\Delta t) = i+1 \mid X(t) = i)$$

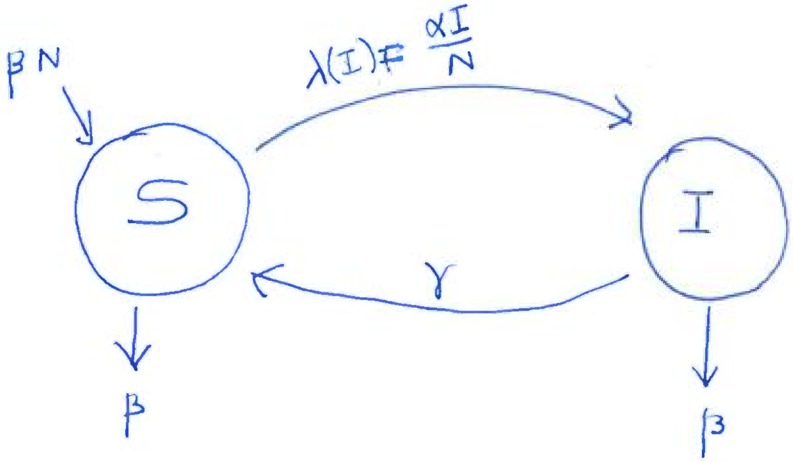
for every $t \in \{0, \Delta t, 2\Delta t, \dots\}$ and for all sequences of states, $0, 1, \dots, i$, whenever both conditional probabilities are well-defined (both will be well-defined $\Leftrightarrow$ $P(X(0)=0, \dots, X(t)=i)$ is not zero).

Probability of transition from state X_t to state $X_{t+\Delta t}$ at time t.

We will only consider the markov chains where the probability of transition does not depend on t.

Stochastic SIS Model (Discrete Time)

It is a Markov chain with finite state space.



Assumptions

- At most one event occurs in the time period Δt (either an infection, birth, death, or recovery) which depends only on the values of the state variables at the current time.
- Population size remains constant $\Rightarrow$ birth and death must occur simultaneously

S, I = random variables for # of susceptible and infective, respectively, in a population of size N.

I is integer-valued with state probabilities

$$P_i(t) = \text{Prob}\{I(t) = i\} \quad \text{for} \quad i \in \{0, 1, \ldots, N\}$$

and $t \in \{0, \Delta t, 2\Delta t, \ldots\}$

Definitions:

$\Pi_i \Delta t = \underbrace{\lambda(i)}_{\text{Force of infection}} \Delta t (N - i)$ = prob. of new infective in time Δt

Note: $i \notin [0, N] \Rightarrow \Pi_i \Delta t = 0$

$(\beta+\gamma)\, i\, \Delta t$ = prob. of recovery or death in time Δt.

Let us assume that N is a very large number with $s+i=N$ (# susceptibles + # infectives $= N$)

$$\Rightarrow \quad \frac{s}{N} = \frac{N-i}{N} = \text{fraction of susceptible individuals}$$

We would like to know the probability of having k individuals in S, for example. Trying to answer questions such as this we require all the theory of Markov chains.

Transition probabilities for the SIS model:

$$P(I(t+\Delta t) = i+1 \mid I(t) = i) = \text{prob. of new infective in time } \Delta t$$

$$= \lambda(i)\, \Delta t\, (N-i)$$

$$P(I(t+\Delta t) = i-1 \mid I(t) = i) = (\beta+\gamma)\, i\, \Delta t$$

(Prob. of recovery or death at time Δt)

Let $P_i(t) = P(I(t+\Delta t) = i \mid I(t) = i)$

Thus,

$$
(2)\quad
\begin{cases}
P_i(t+\Delta t) = P_{i-1}(t)\,\underbrace{\lambda(i-1)\,(N-(i-1))\,\Delta t}_{\text{Prob. New infective}} + P_{i+1}(t)\,\underbrace{(\beta+\gamma)(i+1)\,\Delta t}_{\text{Prob. of recovery or death}} \\
\qquad + P_i(t)\left[1 - \lambda(i)(N-i)\Delta t - (\beta+\gamma)\,i\,\Delta t\right] \\
\qquad\qquad\qquad \underline{\text{for } i = 1, \dots, N} \\
P_0(t+\Delta t) = P_0(t)
\end{cases}
$$

Remark : $P_i(t) = 0$ for $i \notin \{0, 1, \dots, N\}$

Define the vector $p(t) = [P_0(t), P_1(t), \dots, P_N(t)]^T$

$p(t)$ is known as the probability density of I.

We can write (2) in matrix form : $p(t+\Delta t) = T\,p(t)$

where T is known as the $(N+1)\times(N+1)$ transition matrix :

$$
T = \begin{bmatrix}
1 & (\beta+\gamma)\Delta t & 0 & 0 & \cdots & 0 \\
0 & 1-\Pi_1\Delta t-(\beta+\gamma)\Delta t & 2(\beta+\gamma)\Delta t & 0 & & 0 \\
0 & \Pi_1\Delta t & 1-\Pi_2\Delta t-2(\beta+\gamma)\Delta t & 3(\beta+\gamma)\Delta t & & 0 \\
\vdots & \vdots & \vdots & 1-\Pi_3\Delta t-3(\beta+\gamma)\Delta t & & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & 0 & & N(\beta+\gamma)\Delta t \\
0 & 0 & 0 & 0 & & 1-N(\beta+\gamma)\Delta t
\end{bmatrix}
$$

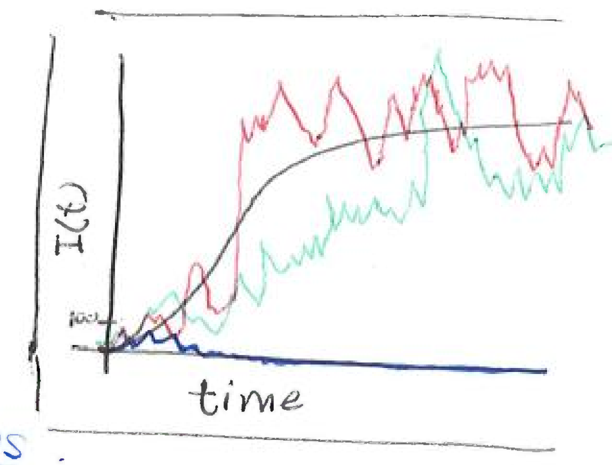
We must be careful here because the entries of T are probabilities. Thus, we must satisfy

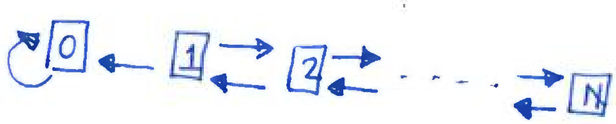
$$\underbrace{\Pi_i}_{\lambda(i)(N-i)\Delta t} \Delta t + (\beta+\gamma)\, i\, \Delta t \leq 1 \qquad (3)$$

Eqn (3) is satisfied if :

$$\begin{cases} N\cdot(\alpha+\beta+\gamma)^2 \Delta t \leq 4\alpha & \text{if } R_0 > 1 \\ (\beta+\gamma) N \Delta t \leq 1 & \text{if } R_0 \leq 1 \end{cases}$$

$$\boxed{R_0 = \frac{\alpha}{\beta+\gamma}}$$





$P_0(t)$ = Probability of no infections

$P_0(t)$ is an absorving state : $\lim_{t\to\infty} P_0(t) = 1$

In fact, $\lim_{t\to\infty} p(t) = (1, 0, \ldots, 0)^T$

for any initial distribution: $p(0) = (P_0(0), \ldots, P_N(0))^T$

- Once there is not infective population no new infections can occur.

$\lim_{t\to\infty} P_i(t) = 0$ for $i \in [1, N]$

If you have infective individuals the prob. that the same amount remains as $t\to\infty$ is 0.