MATH 338: Homework #9

Due: Monday, May 6, 2019

This is the last homework assignment of the semester. Please hand in a **physical copy** to my office by **6 pm on Monday, May 6**$^{\text{th}}$.

1. As discussed in class, the Luria-Delbrück distribution can be generalized to different (deterministic) growth rates for the sensitive and resistant bacteria. Assume, as in class, the following growth laws:

$$\frac{dn}{dt} = \alpha(n+m,t)n$$
$$\frac{dm}{dt} = c\alpha(n+m,t)m$$

   Here n and m denote the number of sensitive and resistant cells, respectively, at time t, and $c > 0$ is a constant.

   (a) Let Z denote the random variable of the number of mutants when the sensitive population is at size N. Find an *approximate* expression for the cumulant generating function of Z, $\psi_Z(s)$.
   Hint: As in class, write it as a sum, and approximate the sum via an integral.

   (b) Your expression in (a) should involve the mutation rate ν. Using a Taylor series calculation, find the leading-order (i.e. first order) terms in ν. Recall that this approximation is valid when ν is small, which we assume here.

   (c) Using your approximation in part (b), find the mean and variance of Z.

2. Consider the chemical reaction network modeling the production and degradation of mRNA:

$$\emptyset \xrightarrow{\alpha} M \xrightarrow{\beta} \emptyset$$

   Recall that the corresponding chemical master equation (CME) is given by

$$\frac{dp_k}{dt} = \alpha p_{k-1} + (k+1)\beta p_{k+1} - \alpha p_k - k\beta p_k,$$

so that the stationary distribution π must satisfy

$$\alpha\pi_{k-1} + (k+1)\beta\pi_{k+1} - \alpha\pi_k - k\beta\pi_k = 0. \tag{1}$$

Here $\pi = (\pi)_{k=0}^{\infty}$ is a distribution vector (note: it is infinite in length). The corresponding solution is given by

$$\pi_k = e^{-\lambda}\frac{\lambda^k}{k!}, \tag{2}$$

where

$$\lambda = \frac{\alpha}{\beta}.$$

Verify this claim. That is, show that π defined by (2) satisfies (1).

3. Consider now a simple model of transcription and translation:

$$\begin{aligned} \emptyset \xrightarrow{\alpha} M &\xrightarrow{\beta} \emptyset \\ M &\xrightarrow{\theta} M + P \\ P &\xrightarrow{\delta} \emptyset \end{aligned}$$

Here P represents the protein translated from M.

(a) Write down the stochiometry matrix Γ for above network.

(b) Assuming mass-action kinetics, write down the propensity functions (i.e. rates) ρ_j^σ, for $j = 1, 2, 3, 4$. Note that the above system contains 4 reactions.

(c) Write down the chemical master equation (CME) for the transcription/translation model. Recall here that

$$k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

is now a vector, as their are two chemical species in the network (M and P).