

## MATH 338: Homework #9

**Due: Monday, May 6, 2019**

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This is the last homework assignment of the semester. Please hand in a **physical copy** to my office by **6 pm on Monday, May 6<sup>th</sup>**.

1. As discussed in class, the Luria-Delbrück distribution can be generalized to different (deterministic) growth rates for the sensitive and resistant bacteria. Assume, as in class, the following growth laws:

$$\begin{aligned}\frac{dn}{dt} &= \alpha(n + m, t)n \\ \frac{dm}{dt} &= c\alpha(n + m, t)m\end{aligned}$$

Here  $n$  and  $m$  denote the number of sensitive and resistant cells, respectively, at time  $t$ , and  $c > 0$  is a constant.

- (a) Let  $Z$  denote the random variable of the number of mutants when the sensitive population is at size  $N$ . Find an *approximate* expression for the cumulant generating function of  $Z$ ,  $\psi_Z(s)$ .  
*Hint:* As in class, write it as a sum, and approximate the sum via an integral.
  - (b) Your expression in (a) should involve the mutation rate  $\nu$ . Using a Taylor series calculation, find the leading-order (i.e. first order) terms in  $\nu$ . Recall that this approximation is valid when  $\nu$  is small, which we assume here.
  - (c) Using your approximation in part (b), find the mean and variance of  $Z$ .
2. Consider the chemical reaction network modeling the production and degradation of mRNA:



Recall that the corresponding chemical master equation (CME) is given by

$$\frac{dp_k}{dt} = \alpha p_{k-1} + (k+1)\beta p_{k+1} - \alpha p_k - k\beta p_k,$$

so that the stationary distribution  $\pi$  must satisfy

$$\alpha\pi_{k-1} + (k+1)\beta\pi_{k+1} - \alpha\pi_k - k\beta\pi_k = 0. \quad (1)$$

Here  $\pi = (\pi)_{k=0}^{\infty}$  is a distribution vector (note: it is infinite in length). The corresponding solution is given by

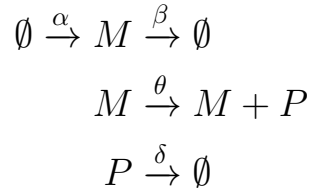
$$\pi_k = e^{-\lambda} \frac{\lambda^k}{k!}, \quad (2)$$

where

$$\lambda = \frac{\alpha}{\beta}.$$

Verify this claim. That is, show that  $\pi$  defined by (2) satisfies (1).

3. Consider now a simple model of transcription and translation:



Here  $P$  represents the protein translated from  $M$ .

- Write down the stoichiometry matrix  $\Gamma$  for above network.
- Assuming mass-action kinetics, write down the propensity functions (i.e. rates)  $\rho_j^\sigma$ , for  $j = 1, 2, 3, 4$ . Note that the above system contains 4 reactions.
- Write down the chemical master equation (CME) for the transcription/translation model. Recall here that

$$k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

is now a vector, as there are two chemical species in the network ( $M$  and  $P$ ).