# MATH 338: Homework #8

## Due: Thursday, April 25, 2019

Solve the below problems related to Markov chains, the Luria-Delbrück distribution, and exponential random variables.

1. This is just the last problem from Exam 2.

   Consider a population of individuals able to produce offspring of the same kind, so that each individual, by the end of its lifetime, will have produced $j$ new offspring with probability $p_j$ for $j = 0, 1, 2, \ldots$, independently of the numbers produced by the other individuals. Clearly then,

   $$0 \leq p_j \leq 1,$$
   $$\sum_{j=0}^{\infty} p_j = 1.$$

   Suppose that there exists ***one*** **individual in the initial generation** (generation 0), and let $X(t)$ denote the size of the $t^{\text{th}}$ generation, for $t = 0, 1, 2, \ldots$. More formally,

   $$X(t) = \sum_{i=1}^{X(t-1)} Z_i$$

   where $Z_i$ denotes the number of offspring of the $i^{\text{th}}$ individual of the $(t-1)^{\text{st}}$ generation, and the $Z_i$ have distribution described by the previous $p_j$.

   (a) Prove that the above construction forms a Markov chain. Describe the state space.

   (b) Find an expression for the transition probability $p_{i,0}$. That is,

   $$p_{i,0} = \mathbb{P}(X(t+1) = 0 \,|\, X(t) = i).$$

   (c) Assume $p_0 > 0$. Using your result from part (b), determine which states are transient and which are recurrent. Note that $p_0 > 0$ is important, and you should provide justification.

(d) Denote the mean and variance of each $Z_i$ as follows:

$$\mu := \mathbb{E}[Z_i] = \sum_{j=0}^{\infty} j p_j$$

$$\sigma^2 := \mathrm{Var}(Z_i) = \sum_{j=0}^{\infty} (j-\mu)^2 p_j.$$

Determine a recurrence relation between $\mathbb{E}[X(t+1)]$ and $\mathbb{E}[X(t)]$. *Hint:* First determine $\mathbb{E}[X(t+1) \mid X(t) = i]$, and use the law of total expectation.

(e) Solve the recurrence relation in part (d) for $\mathbb{E}[X(t)]$.

(f) Assume that $\mu < 1$. Show that the species eventually dies out, i.e. that

$$\pi_0 := \lim_{t\to\infty} \mathbb{P}(X(t) = 0 \mid X(0) = 1) = 1.$$

2. Problem 4.5.2 in the textbook (page 52).

3. In the derivation of the Luria-Delbrück mean and variance for the spontaneous mutation (SM) hypothesis, we assumed that the resistant bacterium grew at the same rate as their sensitive counterparts. In general, this is probably not a realistic assumption, since cells normally need to "give up" something to become resistant. Assume now that **resistant cells divide *half* as often** as the sensitive variants. Under this assumption (and everything else as in class), derive an expression for

$$\frac{\mathbb{E}(Z)}{\mathrm{Var}(Z)},$$

where $Z$ is the number of mutants when the phage is applied. **Please provide the FULL derivation.**

4. Suppose that $\{X_i\}_{i=1}^{N}$ are **independent** random variables, and define

$$Y = \sum_{i=1}^{N} X_i.$$

Find an expression for the cumulant generating function of $Y$ in terms of the cumulant generating functions of the $X_i$.

*Hint:* It may be useful to first work in terms of the moment-generating function

$$\phi(s) = \mathbb{E}(e^{sX}),$$

and to use the relation

$$\psi(s) = \log(\phi(s)).$$

5. The exponential distribution is fundamentally important for continuous-time stochastic processes, as it is the *only* memoryless continuous distribution. That is, if $T$ is exponentially distributed, it satisfies the relationship

$$\mathbb{P}(T > t + s) = \mathbb{P}(T > t)\mathbb{P}(T > s). \tag{1}$$

   (a) Verify equation (1) for any exponential distribution.

   (b) The functional relationship in (1) takes the form

$$G(t + s) = G(t)G(s) \tag{2}$$

   We now show that the only continuous (actually, you only need right-continuity) solution of (2) is an exponential. Show first that for any **positive integer** $n$,

$$G(n) = G(1)^n.$$

   (c) Similarly to (b), conclude the same for all **negative integers**.

   (d) Show that for any **positive integer** $n$,

$$G\left(\frac{1}{n}\right) = G(1)^{\frac{1}{n}}.$$

   (e) Show that your previous results imply that for **any rational number** $x$,

$$G(x) = G(1)^x.$$

   (f) The last part requires a bit of mathematical machinery, so I will just tell you that continuity and part (e) imply that

$$G(x) = G(1)^x$$

for **all real numbers** $x$. Show that

$$G(x) = e^{-\lambda x}$$

for some $\lambda > 0$. Find the value of $\lambda$, and make sure that you know it is strictly positive (remember, $G$ should be decreasing, since it represents $G(x) = \mathbb{P}(T > x)$). This then completes the proof.