MATH 338: Homework #6

Due: Thursday, March 28, 2019

Solve the below problems concerning Markov chains.

1. Consider the following model for genetics of inbreeding for a diploid species with 2 alleles (A and a) at a fixed locus. Consider a random population in which two individuals are randomly mated. Then, in the next generation, two of their offspring are randomly mated (here random means randomly sampled). Denote the state at time t ($X(t)$) the offspring chosen to mate. This is then a pair of genotypes (one for each parent selected), which we list of the form, for example, $\{AA, aa\}$. That is, $X(t)$ takes one of the **six** possible values

   (i) $\{AA, AA\}$
   (ii) $\{AA, Aa\}$
   (iii) $\{Aa, Aa\}$
   (iv) $\{Aa, aa\}$
   (v) $\{AA, aa\}$
   (vi) $\{aa, aa\}$.

   Note that the order is unimportant. Find the state transition matrix for this Markov chain.

2. Consider the randomly perturbed difference equation

$$X(t+1) = f(X(t), \xi(t+1),$$

   where

   (i) $\{\xi(t)\}_{t\in\mathbb{Z}_+}$ is a sequence of random inputs (and hence random variables) that are identically and independently distributed, and
   (ii) $X(0)$ is independent of $\{\xi(t)\}_{t\in\mathbb{Z}_+}$.

   Show that $X(t)$ is Markov, and find an expression for the one-step transition probabilities $p_{i,j}$:

$$p_{i,j} = \mathbb{P}(X(t+1) = j | X(t) = i).$$

3. Let A be the probability transition matrix of a Markov chain. Show that if for some positive integer, A^r has all positive entries, then so does A^n, for all integers $n \geq r$.

4. Recall that a DNA nucleotide has any one of the four values $\{A, C, T, G\}$. Fix $0 < \alpha < 1/3$. A standard model for a mutational change of the nucleotide at a specific location is a Markov chain that supposes in going from period to period (here we assume **discrete** time units) the nucleotide does not change with probability $1 - 3\alpha$, and if it does change then it is equally likely to change to any of the other 3 possible values.

   (a) Denote by $X(t)$ the current base (one of $\{A, C, T, G\}$) at a specific location at time t. Find the transition probabilities for the Markov chain $X(t)$.

   (b) Find $p_{i,j}^{(t)}$, for any $t \geq 0$, and any $i, j \in \{A, C, T, G\}$.

   (c) What is the long-time behavior of the distribution of $X(t)$?

5. Show that the expected value of the Moran Markov chain is constant in time.

6. For a Markov chain with state space $\mathcal{E} = \{1, 2, 3, 4\}$, the state transition matrix is

$$A = \begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/12 & 5/12 & 1/2 & 0 \\ 0 & 1/6 & 1/2 & 1/3 \\ 0 & 0 & 1/3 & 2/3 \end{pmatrix}$$

   Assume that the initial distribution satisfies

$$\rho(0) = (1/4, 1/6, 1/3, 1/4).$$

   Note for the below questions, you are advised to calculate matrix powers. You may use any software of your choosing, but the answers to the below should be **numbers**. Note that $X^2(t) = (X(t))^2$.

   (a) Find $\mathbb{P}(X(3) = 3)$.

   (b) Find $\mathbb{E}[X^2(5) \mid X(4) = 1]$.

   (c) Find $\mathbb{E}[X^2(3)]$.