MATH 338: Homework #5

Due: Thursday, March 14, 2019

Solve the below problems concerning stochastic processes and Markov chains.

1. Consider a three-state (that is, $\mathcal{E} = \{1, 2, 3\}$) Markov chain with transition probability matrix

$$\begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}.$$

   (a) Write down the *state transition diagram* for this process.

   (b) Assume that $\mathbb{P}(X(0) = 1) = 1$. Find the probability of the following sample path:

$$\mathbb{P}(X(0) = 1, X(1) = 2, X(2) = 2).$$

   (c) Again assume that $\mathbb{P}(X(0) = 1) = 1$. Find the probability of the following sample path:

$$\mathbb{P}(X(0) = 2, X(1) = 2, X(2) = 2).$$

   (d) Again assume that $\mathbb{P}(X(0) = 1) = 1$. Find the probability of being at state 1 at time $t = 2$:

$$\mathbb{P}(X(2) = 1).$$

2. Find transition probabilities for the following Markov chains.

   (a) Urn I and Urn II both contain N marbles. Of the $2N$ marbles, half are white and half are red. At each step, one marble is selected at random from each urn and the marbles are interchanged. $X(t)$ represents the number of white marbles in the first urn at the t^{th} step.

   (b) Now assume Urn I and Urn II contain a total of M marbles. Let the state be the number of marbles in Urn I. At each step, proceed as follows: if the state is k, select Urn I with probability k/M, otherwise select Urn II; then select a marble from Urn I with probability p and from Urn II with probability $q = 1 - p$, and place the marble in the selected urn.

3. Consider the Wright-Fisher chain with $N = 10$. Suppose that $\mathbb{P}(X(0) = 2) = 1$. Find $\mathbb{P}(X(2) = 10)$.

4. Consider the Moran model with $N = 10$ and $\mathbb{P}(X(0) = 1) = 1$. Find $\mathbb{P}(X(3) = 0)$ and $\mathbb{P}(X(3) = 1)$.

5. Calculate the expected lifetime of an individual in the Moran model.

6. Recall that the simple random takes the form

$$X(t) = X(0) + \sum_{i=1}^{t} \xi_i,$$

where ξ_i are independently identically random variables such that

$$\mathbb{P}(\xi_i = 1) = p, \qquad \mathbb{P}(\xi_i = -1) = q = 1 - p.$$

Derive an expression for the expected value of the position of the random walk at time t (i.e. $\mathbb{E}(X(t))$). Note that your answer should depend on the initial distribution, time t, and p.

7. Exercise 4.1.7 in Chapter 4 of the online notes (page 20).