MATH 338: Homework #4

Due: Tuesday, March 5, 2019

Solve the below problems concerning selection models and nonlinear difference equations.

1. This problem is a numerical example of the selection model presented in class (and in Section 3.4). The main point is to get familiar with working with difference equations that cannot be solved analytically, as well as to get practice interpreting the defined quantities.

   Suppose that $w_{AA} = 0.6, w_{Aa} = 0.7$, and $w_{aa} = 0.4$. Consider an initial population (generation 0) which is at birth in Hardy-Weinberg equilibrium with $f_A(0) = 0.2$.

   (a) Find the genotype frequencies of the initial population.

   (b) Find the probability that a randomly selected individual survives to reproductive maturity in the initial generation.

   (c) Find the probability $p_{Aa}(0)$ that an individual who survives to reproductive maturity has genotype Aa.

   (d) What is the frequency $f_A(1)$ of allele A in the next generation?

2. Consider a locus with two alleles A and a in a model in which selection acts. The probability an individual survives to reproductive maturity is 0.6 given it is AA, 0.7 given it is Aa, and 0.4 given it is aa. Let $f_A(t)$ denote the frequency of A in generation t of newborns. Assume $f_{AA}(0) = 0.16$, $f_{Aa}(0) = 0.48$, and $f_{aa}(0) = 0.36$.

   (a) Write down a difference equation for $f_A(t)$ of the form $f_A(t+1) = \phi(f_A(t))$.

   (b) Make a sketch of $\phi(p)$, for $0 \leq p \leq 1$. Be as detailed as possible.

   (c) Find $\lim_{t\to\infty} f_A(t)$, and justify using the sketch from part (b). That is, include cobwebbing.

   (d) What is $\lim_{t\to\infty} f_a(t)$?

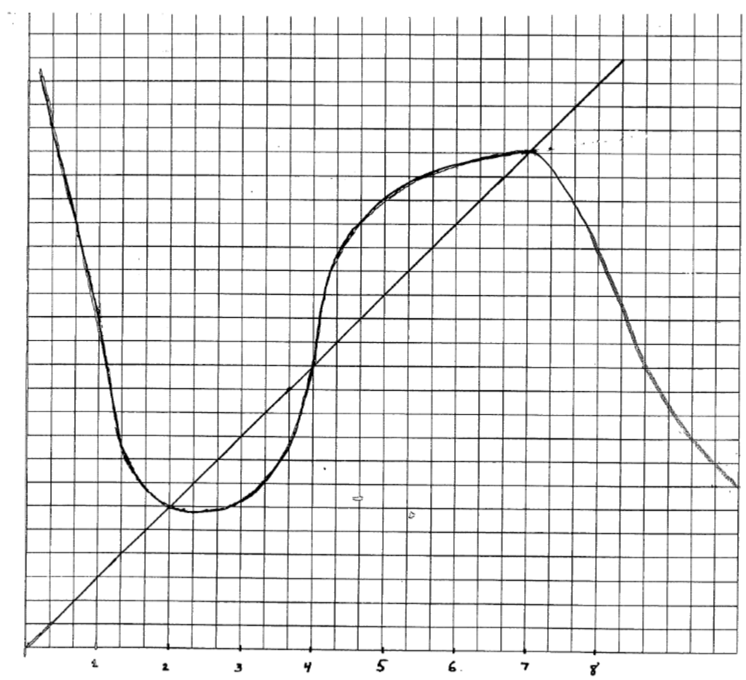
   (e) What is the probability that an individual of generation 0 survives?

(f) What is the probability that an individual chosen randomly from the survivors of generation 0 is genotype AA?

3. Consider the first-order difference equation

$$f(t+1) = \phi(f(t)), \tag{1}$$

where the function ϕ is plotted below:



(a) Identify the fixed points of the difference equation (1). Label each as either stable or unstable. Provide (brief) justification.

(b) Let $f(t)$ be the solution with $f(0) = 1$. Trace the cobweb evolution on the graph of ϕ, and provide a plot of $f(t)$ vs. t on a separate plot. Does $\lim_{t\to\infty} f(t)$ exist? if so, what is it?

(c) Find the **basis of attraction** for each stable fixed point. The basin of attraction is set of all initial conditions whose trajectories limit to that fixed point.

4. The dynamics of the selection model were shown to be described by the first-order nonlinear difference equation

$$f(t+1) = \phi(f(t)),$$

where

$$\begin{aligned} f(t) &= f_A(t), \\ \phi(p) &= \frac{w_{AA}p^2 + w_{Aa}p(1-p)}{W(p)}, \\ W(p) &= w_{AA}p^2 + 2w_{Aa}p(1-p) + w_{aa}(1-p)^2. \end{aligned}$$

As $W(f_A(t))$ represents the probability that a randomly selected individual from the infant population survives to reproduction, it can be interpreted as the *mean fitness of generation* t. Show that for the one locus/two allele selection model, the mean fitness always increases from generation to generation, i.e.

$$W(f_A(t+1)) > W(f_A(t)), \quad \text{if } f_A(t) \text{ is not a fixed point.}$$

5. Suppose it is known that $w_{AA} = w_{Aa} = w$, and $w > w_{aa}$ in the selection model. Note that this case was NOT covered in class. Determine $\lim_{t\to\infty} f_A(t)$ by analyzing the shape of ϕ and cobwebbing.

6. The purpose of this problem is to model the situation in which both selection and mutation can occur. Assume that the selection coefficients w_{AA}, w_{Aa}, and w_{aa} are given and interpret them as survival probabilities. Assume in addition that in the process of reproduction,

   (i) A mutates to a with probability u, and
   (ii) a mutates to A with probability v.

   (a) With $f_A(t)$ and $p_A(t)$ defined as in class (and in the text), argue that the between generation evolution is given by

   $$f_A(t+1) = (1-u-v)p_A(t) + v.$$

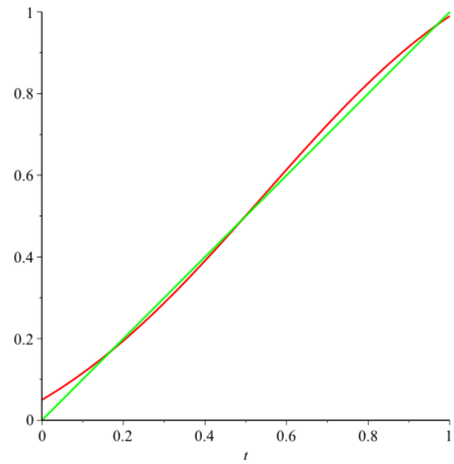
   (b) Now use expressions derived in class for $p_A(t)$ in terms of $f_A(t)$ and the selection coefficients to find a nonlinear difference equation

   $$f_A(t+1) = \psi(f_A(t))$$

   (c) Suppose now that $w_{Aa} = 0.5, w_{AA} = 0.7$, and $w_{aa} = 0.8$, and $u = 0.05, v = 0.01$. In this case, the model for $f_A(t)$ is

   $$\begin{aligned} f_A(t+1) &= \psi(f_A(t)), \\ \psi(p) &= \frac{2.13p^2 + 4.4p + 0.4}{5p^2 - 6p + 8} \end{aligned}$$

The graph of ψ is shown below:



The fixed points are approximately 0.490.0.169, and 0.967.

(d) If $f_A(0) = 0.55$, what is $\lim_{t\to\infty} f_A(t)$? What would this limit be if there were no mutation, only selection, with the same selection coefficients?

(e) If $f_A(0) = 0.4$, what is $\lim_{t\to\infty} f_A(t)$? What would this limit be if there were no mutation, only selection, with the same selection coefficients?