MATH 338: Homework #3

Due: Thursday, February 21, 2019

Solve the below problems concerning the evolution of population genetics in the absence of selection.

1. In class, we showed that the evolution of the allele frequency for a gene on the X chromosome of the female population in a dioecious species satisfying assumptions A1, A2, A3, and A5 satisfied the second-order linear difference equation

$$x(t+1) = \frac{1}{2}\left(x(t) + x(t-1)\right) \tag{1}$$

   for all $t \in \mathbb{N}$. Show that solutions of (1) satisfy **superposition**, i.e. that if $x_1(t)$ and $x_2(t)$ are two solutions of (1), then so is

$$x(t) := c_1 x_1(t) + c_2 x_2(t),$$

   for any $c_1, c_2 \in \mathbb{R}$.

2. For the simple model of mutation with no selection (the rest of A1-A5 are assumed), the difference equation describing the dynamics of the frequency of allele A is

$$f_A(t+1) = v + (1-u-v) f_A(t). \tag{2}$$

   Recall that u and v denote the probability of the gametes mutating from A to a, and a to A, respectively.

   (a) Find the solution of (2) with respect to the initial frequency $f_A(0)$ (*Hint:* See the Appendix of Chapter 3 for a discussion of difference equations).

   (b) If $u, v > 0$, do alleles A and a both persist in the population forever? (*Hint:* What is $\lim_{t\to\infty} f_A(t)$?).

   (c) What happens if $v = 0$, but $u > 0$? Provide justification.

   (d) Describe the dynamics in the case $u = v = 1$.

3. We now generalize the scenario of Problem #2 for a **dioecious** species, with a locus located on the X chromosome. Assume the following:

(i) A and a are the two alleles on the X chromosome.

(ii) For males transmitting alleles to an offspring, A mutates to a with probability 0.2, while a mutates to A with probability 0.3.

(iii) For females transmitting alleles to an offspring, A mutates to a with probability 0.1, while a mutates to A with probability 0.05.

Assume non-overlapping generations, infinite populations, and random mating. Let $f_A^m(t)$ and $f_A^f(t)$ denote the frequency of A in generation t for males and females, respectively.

(a) Express $f_A^f(t+1)$ in terms of $f_A^m(t)$ and $f_A^f(t)$.

(b) Repeat part (a) for $f_A^m(t+1)$.

(c) Derive a second-order difference equation for $f_A^f(t)$ (note: you do NOT need to solve it)

4. Consider a locus with two alleles A and a in a monecious population. In its home territory on the mainland, the bird population is and stays in Hardy-Weinberg equilibrium with $f_A = 0.3$. There is also a colony population on an island.

   Assume: for each t, 20% of generation $t+1$ on the island are immigrants from the home territory; they are children of random matings of parents on the mainland. The remaining 80% of generation $t+1$ on the island are the result of random mating among generation t parents from the island. Assume non-overlapping generations and the infinite population assumption.

   (a) Let $g_A(t)$ be the frequency of allele A in generation t *on the island*. Find a difference equation for $g_A(t)$ and solve it, assuming $g_A(0) = 0.5$.

   (b) Find $\lim_{t\to\infty} g_A(t)$, and explain intuitively why we should see this limiting value.

5. Overlapping generations were introduced on Thursday (2/14), and will be discussed further on Tuesday (2/19). Consider the following variation, where newborns require ***two*** seasons to mature (i.e. do NOT mate in the first season they are born). Assume mating occurs at times $h, 2h, \ldots$, and in each season a fraction h of the population is replaced (as before). Note that newborns have the same probability as older individuals to be removed, so

that they may not survive until sexual maturity. Conditions A1, A3, A4, and A5 are still assumed.

Let $f_A(t)$ be the A allele frequency in the population who are at least 1 season old (i.e. born at $t-h$ or before), and $g_A(t)$ be the frequency of allele A in the population born at t. Derive a set of difference equations for $f_A(t)$ and $g_A(t)$, reduce to one equation for $f_A(t)$, and solve.