MATH 338: Homework #1

Due: Thursday, February 7, 2019

Solve the below problems reviewing basic probably theory and introductory modeling.

1. Consider a probability triple $(\Omega, \mathcal{F}, P)$, and let A be any event (i.e. $A \in \mathcal{F}$). Show that

$$P(A^c) = 1 - P(A).$$

   Recall that $A^c = \Omega \setminus A = \{x \in \Omega \,|\, x \notin A\}$.

2. Three coins are in a box. Two are fair and one is loaded; when flipped, the loaded coin comes up heads with probability 2/3. A coin is chosen by **random sampling** (please see Section 2.1.2 in the notes for the precise meaning here) from the box and flipped. What is the probability that it comes up heads? Given that it comes up heads, what is the conditional probability that it is the loaded coin?

3. The following is related to gene expression, but is translated into a purely mathematical problem. For those who know a bit of genetics (and we will cover this soon), I hope you are able to see the connection.
   Consider taking a sequence of length 4 such that each element belongs to the set $\{A, T, G, C\}$. Suppose that each sequence is equally likely.

   (a) Describe the probability space. That is, identify the sample space Ω, the set of events $\mathcal{F}$, and the probability measure P.

   (b) Find the probability that the selected sequence contains **exactly** two A elements.

4. Consider now an extension of Problem #2, where **two** sequences of length 4 are aligned as follows:

$$x_1\, x_2\, x_3\, x_4$$
$$y_1\, y_2\, y_3\, y_4,$$

   where $x_i, y_i \in \{A, T, G, C\}$. You should think of this as a sequence of 4 base pairs in a segment of DNA.

(a) Assume the selection of the x-sequence and the selection of the y-sequence are independent, and that both are random samples of size 4 from the alphabet $\{A, T, G, C\}$. Construct a probability space for the aligned pair of sequences.

(b) What is the probability that both the x-sequence and the y-sequence begin with A? What is the probability that x_1 and x_2 are equal? What is the probability that an A is aligned with A exactly twice? What is the probability that $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4$?

5. Let X be an exponential random variable with parameter λ. For $k = 1, 2, 3, \ldots$, let

$$Y = k, \qquad \text{if} \qquad k - 1 < X < k.$$

Prove that Y is a geometric random variable with parameter $p = 1 - e^{-\lambda}$.

6. Let $Y = e^X$, where X is normally distributed with mean 0, and variance 1 ($X \sim N(0, 1)$). Find the density function of Y.

7. In class, we studied the "death" process for an individual. We now extend the analysis to an *independent* population of N individuals. For $i = 1, 2, \ldots, N$, construct the random variables

$$X_i^{(t)} = \begin{cases} 1, & \text{if individual i is alive at time t;} \\ 0, & \text{if individual i is dead at time t;} \end{cases}$$

As in class, assume that

$$P(X_i^{(t)} = 1) = e^{-\mu t},$$

where $\mu > 0$ is constant and identical for all N individuals. Note that this is just the probability that individual i is alive at time t_1. Assuming that the $X_i^{(t)}$ are independent for all time t, define the random variable Y_t as

$$Y_t = \sum_{i=1}^{N} X_i^{(t)}.$$

(a) Describe, in words, what Y represents.

(b) What is the distribution of Y?

(c) Show that, for any $t_1, t_2 \geq 0$,

$$E(Y_{t_1+t_2}) = E(Y_{t_1}) e^{-\mu t_2}.$$