Final Exam Review Problems

For details on exam coverage and a list of topics, please see the class website. Note that this is a set of review problems, and NOT a practice exam. Most problems below concern material covered after Exam 2. Please see the review problems for Exams 1 and 2 for a more detailed review of material from earlier in the course.

1. Consider a chemical reaction network with master equation

$$\frac{dp_k}{dt} = \sum_{j=1}^{m} \rho_j^\sigma(k-\gamma_j)p_{k-\gamma_j} - \sum_{j=1}^{m} \rho_j^\sigma(k)p_k$$

Suppose that the initial distribution satisfies

$$\sum_{k\in\mathbb{Z}_+^n} p_k(0) = 1$$

(That is, the initial distribution is really a distribution). Show that for any time $t \geq 0$,

$$\sum_{k\in\mathbb{Z}_+^n} p_k(t) = 1$$

That is, that the solutions of the CME are always a probability distribution, assuming that the initial conditions were.

2. Assume that mRNA is produced in "bursts" of $r > 1$ transcripts at a time. Further, as before, mRNA degrades.

   (a) Write down a system of chemical reactions describing the above phenomena.

   (b) Find the stochiometry matrix of the network.

   (c) Assuming mass-action kinetics, write down the propensity functions.

   (d) Write the CME for the bursting model.

3. A population of 10 individuals is divided into two types, A and a. Let $X(t)$ denote the number of type As at step t. There is a separate reservoir of individual in which the frequency of A is always 0.4. The population evolves as follows: with probability 0.1 an individual is removed at random from the population of 10 and replaced by a randomly drawn individual from the reservoir; with probability .9 the population is updated as in the Moran model, that is, two individuals are drawn at random with replacement and the second is replaced by a copy of the first.
Find

$$p_{i,i-1} = \mathbb{P}(X(t+1) = i-1 \,|\, X(t) = i),$$

for $i = 1, 2, \ldots, 10$.

4. State what the infinite population assumption says about the frequency of genotypes among children of a mating population.

5. Let A be the state transition matrix for a Markov chain evolving on the state space $\{0, 1, 2, 3, 4\}$.

$$A = \begin{pmatrix} 1/6 & 5/6 & 0 & 0 & 0 \\ 1/6 & 1/2 & 1/3 & 0 & 0 \\ 0 & 2/3 & 1/6 & 1/6 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 2/3 \end{pmatrix}$$

You can also compute the following powers of A:

$$A^2 = \begin{pmatrix} 1/6 & 5/9 & 5/18 & 0 & 0 \\ 1/9 & 11/18 & 2/9 & 1/18 & 0 \\ 1/9 & 4/9 & 11/36 & 1/12 & 1/18 \\ 0 & 2/9 & 1/6 & 5/18 & 1/3 \\ 0 & 0 & 1/9 & 1/3 & 5/9 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0.12 & 0.60 & 0.23 & 0.05 & 0 \\ 0.12 & 0.55 & 0.26 & 0.05 & 0.02 \\ 0.09 & 0.52 & 0.23 & 0.10 & 0.06 \\ 0.04 & 0.22 & 0.20 & 0.23 & 0.31 \\ 0 & 0.07 & 0.13 & 0.32 & 0.48 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0.12 & 0.56 & 0.25 & 0.05 & 0.02 \\ 0.11 & 0.55 & 0.24 & 0.07 & 0.03 \\ 0.10 & 0.49 & 0.24 & 0.09 & 0.08 \\ 0.04 & 0.27 & 0.18 & 0.22 & 0.29 \\ 0.01 & 0.12 & 0.15 & 0.29 & 0.42 \end{pmatrix}$$

At time $t = 0$,

$$(\mathbb{P}(X(0) = 0), \mathbb{P}(X(0) = 1), \mathbb{P}(X(0) = 2), \mathbb{P}(X(0) = 3), \mathbb{P}(X(0) = 4)) = \left(0, \frac{2}{3}, \frac{1}{6}, 0, \frac{1}{6}\right).$$

(a) Draw a state transition diagram for this chain.

(b) Is the chain irreducible? Why or why not?

(c) Find $\mathbb{P}(X(8) = 3 \mid X(4) = 1)$.

(d) Compute $\mathbb{P}(X(3) = 4)$.

(e) Find $\mathbb{E}(X^2(4) \mid X(3) = 3)$.

(f) Find $\lim_{t\to\infty} \mathbb{P}(X(t) = 3)$.

6. Suppose X takes on each of the values $1, 2$ and 3 with probability $1/3$.

   (a) Find the moment-generating function of X.

   (b) Find the cumulant-generating function of X.

   (c) Using your result from part (b), compute the mean and variance of X.

7. Consider a population of individuals able to produce offspring of the same kind, so that each individual, by the end of its lifetime, will have produced j new offspring with probability p_j for $j = 0, 1, 2, \ldots$, independently of the numbers produced by the other individuals. Clearly then,

$$0 \le p_j \le 1,$$
$$\sum_{j=0}^{\infty} p_j = 1.$$

Suppose that there exists ***one*** **individual in the initial generation** (generation 0), and let $X(t)$ denote the size of the t^{th} generation, for $t = 0, 1, 2, \ldots$. More formally,

$$X(t) = \sum_{i=1}^{X(t-1)} Z_i$$

where Z_i denotes the number of offspring of the i^{th} individual of the $(t-1)^{\text{st}}$ generation, and the Z_i have distribution described by the previous p_j.
Let

$$\mu := \mathbb{E}[Z_i] = \sum_{j=0}^{\infty} j p_j$$

In HW #8, we showed that if $\mu < 1$, then

$$\lim_{t\to\infty} \mathbb{P}(X(t) = 0 \,|\, X(0) = 1) = 1,$$

i.e. that the population dies out with probability 1. Assume now that $\mu > 1$.

(a) Recall that $\pi_0 = \mathbb{P}(\text{population eventually dies out})$, since 0 is an absorbing state. Assume $X(1) = j$. Find the following probability:

$$\mathbb{P}(\text{population eventually dies out} \,|\, X(1) = j)$$

(b) By conditioning on $X(1)$, find an equation satisifed by π_0. *Note:* It can be shown that π_0 is the smallest positive number satisfying the correct equation for this part.

(c) Suppose that $p_0 = 1/2, p_1 = 1/4, p_2 = 1/4$. Find π_0.

(d) Suppose that $p_0 = 1/4, p_1 = 1/4, p_2 = 1/2$. Find π_0.

8. In class, when discussing the Luria-Delbrück distribution, we assumed that the resistant cells grew deterministically:

$$Y_n^N = \frac{N}{n} X_n,$$

or

$$Y_n^N = \left(\frac{N}{n}\right)^c X_n,$$

in the case of relative growth rate c. Now assume that the resistant mutants grow stochastically, so that

$$Y_n^N = K_n^N X_n,$$

where K_n^N is the **random** size of a mutant population at phage application first appearing when there are n sensitive cells. That is, K_n^N is a random variable. All other dynamics are assumed identical to the case discussed in class.

(a) Find an expression for the cumulant generating function of Z, the number of mutants when the phage is applied. *Note:* This will depend on the statistical properties of the random variable K_n^N.

(b) Find the mean and variance of Z.

9. Consider the annihilation of a species A via dimerization:

$$M + M \xrightarrow{\alpha} \emptyset$$

(a) Write down the chemical master equation for this reaction.

(b) What do you expect the stationary distribution π of this reaction network to be?

(c) Prove your claim from part (b).