Exam 2 Review Problems

For details on exam coverage and a list of topics, please see the class website. Note that this is a set of review problems, and NOT a practice exam.

1. Consider a Markov chain $\{X(t)\}$ evolving in the state space $\{0,1,2,3,4\}$. We give its state transition matrix and some of its powers:

$$A = \begin{pmatrix} \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \qquad A^2 = \begin{pmatrix} 1/6 & 5/9 & 5/18 & 0 & 0 \\ 1/9 & 11/18 & 2/9 & 1/18 & 0 \\ 1/9 & 4/9 & 11/36 & 1/12 & 1/18 \\ 0 & 2/9 & 1/6 & 5/18 & 1/3 \\ 0 & 0 & 1/9 & 1/3 & 5/9 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0.12 & 0.60 & 0.23 & 0.05 & 0 \\ 0.12 & 0.55 & 0.26 & 0.05 & 0.02 \\ 0.09 & 0.52 & 0.23 & 0.10 & 0.06 \\ 0.04 & 0.22 & 0.20 & 0.23 & 0.31 \\ 0 & 0.07 & 0.13 & 0.32 & 0.48 \end{pmatrix} \qquad A^4 = \begin{pmatrix} 0.12 & 0.56 & 0.25 & 0.05 & 0.02 \\ 0.11 & 0.55 & 0.24 & 0.07 & 0.03 \\ 0.10 & 0.49 & 0.24 & 0.09 & 0.08 \\ 0.04 & 0.27 & 0.18 & 0.22 & 0.29 \\ 0.01 & 0.12 & 0.15 & 0.29 & 0.42 \end{pmatrix}$$

   At time $t = 0$, let the initial distribution be given by

$$\rho(0) = (0, 2/3, 1/6, 0, 1/6).$$

   (a) Draw the state transition diagram for the above Markov chain.
   (b) What are the communication classes of the Markov chain?
   (c) Is the chain irreducible?
   (d) Find $\mathbb{P}(X(0) = 1, X(1) = 2, X(2) = 2, X(3) = 3, X(4) = 2)$.
   (e) Compute $\mathbb{P}(X(8) = 3 \,|\, X(4) = 1)$.
   (f) Find $\mathbb{P}(X(2) = 2)$.
   (g) Find $\mathbb{E}(X(3) \,|\, X(0) = 4)$.
   (h) What happens to the distribution of $X(t)$ as $t \to \infty$?

2. Let A be a stochastic matrix. Show that A^t is stochastic for all $t \in \mathbb{Z}_+$.

3. Suppose that in the Moran model, if a type A is chosen to reproduce, its offspring mutates from A to a with probability u, and if a type a reproduces its offspring mutates to type A with probability v.

   (a) Find the transition probabilities for this Markov chain.
   (b) Is the chain irreducible? Provide justification.

4. In HW#5, we derived the Wright-Fisher model with mutation. Recall that the conditional distribution of $X(t+1)$ given $X(t) = i$ is binomial with parameters $n = 2N$ and

$$p = v + (1 - u - v)\frac{i}{2N}.$$

   Here u is the probability that A mutates to a, and v is the probability that a mutates to A. Let

$$Y(t) := \frac{X(t)}{2N}$$

   be the frequency of A. Show that

$$\mathbb{E}(Y(t+1)) = v + (1 - u - v)\mathbb{E}(Y(t)).$$

   From this compute

$$\lim_{t\to\infty} \mathbb{E}(Y(t)).$$

5. A flea moves around the vertices of a triangle in the following manner: whenever it is at vertex i¡ it moves to its clockwise neighbor vertex with probability p_i and to the counterclockwise neighbor with probability $q_i = 1 - p_i$, for $i = 1, 2, 3$.

   (a) Find the state transition matrix for this Markov chain.

   (b) Is it irreducible and aperiodic?

   (c) Find the proportion of time that the flea is at each vertex.

6. Consider the neutral (i.e. no mutation) Moran model. Find the variance of this process, given $X(0) = i$:

$$\operatorname{Var}(X(t) \mid X(0) = i).$$

   Hint: As usual, it will be useful to condition on the previous value of $X(t)$. You may also want to use the *law of total variance*: for random variables X and Y defined on the sample probability space, and if the variance of Y is finite, then

$$\operatorname{Var}(Y) = \mathbb{E}[\operatorname{Var}(Y \mid X)] + \operatorname{Var}(\mathbb{E}[Y \mid X]).$$

7. Let $\{X(t)\}$ be a Markov chain on state space $\{0, \pi/2, \pi\}$ with state transition matrix

$$A = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 5/9 & 1/3 & 1/9 \\ 0 & 1/5 & 4/5 \end{pmatrix}.$$

   Suppose that

$$\begin{aligned} \mathbb{P}(X(0) = 0) &= 1/10 \\ \mathbb{P}(X(0) = \pi/2) &= 4/5 \\ \mathbb{P}(X(0) = \pi) &= 1/10. \end{aligned}$$

   Find $\mathbb{E}[(\sin(X(t))]$.

8. (Possibly hard) For the Moran model without mutation, we saw that the fixation occured with probability 1. Let T_i be the expected time before fixation, given that $X(0) = i$. Find

$$\mathbb{E}[T_i],$$

   i.e. the expected time before fixation, given that the process starts with i A alleles.