MATH 336: Homework #5

Due: Tuesday, March 1, 2016

Solve the below problems concerning differential equations. A (possibly improper) subset of them will be graded. All calculations should be done analytically, unless marked with an (M). (M) problems require the use of MATLAB. ES denotes the online lecture notes.

1. (20 points) (M) In this problem, I would like you to plot the direction field (i.e. vector field) and solution curves for a specific chemostat.

   Suppose the reactor has a volume of $2m^3$, and a feed and effluent flow rate of $1m^3/s$. The feed tank is held at a constant nutrient concentration of $3g/m^3$, which is measured to grow according to Michaelis-Menten kinetics with a maximum growth rate of $1/s$, and half-saturation concentration of $0.5g/m^3$.

   Use the data given to complete the following MATLAB code, which plots both the direction field as well as some solution curves in the (N, C) plane. It consists of **two** *m-files*, one with the main script (say *hw5No1.m*), and the other with the right-hand side of the ODE (analogously to HW #3). The main script outline is below:

```
%Constants defined for specific chemostat reactor
V=
kMax=
F=
C0=
km=

%Non-dimensional constants and steady states (for plotting)
%alpha1 and alpha2 will be defined in terms of the above constants
alpha1=
alpha2=
N1_steady=0;
C1_steady=alpha2;
N2_steady=alpha1*(alpha2-1/(alpha1-1));
C2_steady=1/(alpha1-1);

%These are just for aligning everything in the plot, so ignore.
N_len=N2_steady+6;
C_len=C1_steady+1;
```

```
%Define the vector field
[N,C]=meshgrid(0:0.5:N_len,0:0.5:C_len);
dN=alpha1*(C./(1+C)).*N-N; %dN/dt right-hand side
dC= %dC/dt right-hand side (add code here!)
L=sqrt(dN.^2+dC.^2);
%Normalize vectors to only observe direction (not magnitude)
dN=dN./L;
dC=dC./L;
figure(1)
quiver(N,C,dN,dC,'r'); %Actual plot command for vector (direction) field
hold on;
%Plotting steady states
plot(N1_steady,C1_steady,'*k','LineWidth',3);
plot(N2_steady,C2_steady,'*k','LineWidth',3);

%Now solve the ode and plot the trajectories from t=0 to t=50 (arbitrary)
N0_vec=[0.5;4;8;11];
C0_vec=[1;4;7];
t0=0;
tF=50;

for i=1:length(N0_vec)
    for j=1:length(C0_vec)
        initials=[N0_vec(i),C0_vec(j)];
        [T,NC]=ode45(@rhsHw5No1,[t0 tF],initials,[],alpha1,alpha2);
        %First column of NC vector is N, second is C (T is time vector which we...
        %.. don't need)
        N_vec=NC(:,1);
        C_vec=NC(:,2);
        plot(N_vec,C_vec,'-b','LineWidth',2);
        plot(N0_vec(i),C0_vec(j),'xb','LineWidth',2); %Also plot initial conditions
    end
end

%Lastly, format the graph
axis([0 16 0 7]);
xlabel('N');
ylabel('C');
title('Chemostat phase portrait and direction field');
```

Note that there a number of statements that are blank, which need to be completed for the code to run. The additional **function** *m-file rhsHw5No1.m* is also provided below, but requires some work as well. Remember that both *m-files* must be in the same directory to run.

```
function dNCdt=rhsHw5No1(t,NC,alpha1,alpha2)

    N=NC(1);  %First component corresponds to bacteria
    C=NC(2);  %Second component corresponds to nutrient

    %The first component corresponds to the equation for dN/dt, while the second is for dC/dt
    dNCdt=[alpha1*(C./(1+C).*N)-N;];
end
```

Attach your plot and answer the following:
Does the output of the code agree with the theoretical results from class? What type of steady state (node, spiral, saddle, etc. and include stability) do you observe for both steady states?

2. (10 points) (ES, p.130-131, #2, part (a)) Problem 2 in the ODE2 section of the notes (end of chapter 2), **part (a) only.**

3. (20 points) (ES, p.131, #3) Problem 3 in the ODE2 section of the notes (end of chapter 2).

4. (20 points) (ES, p.132, #4, parts (a) and (d)) Problem 4 in the ODE2 section of the notes (end of chapter 2), **parts (a) and (d) only.** Note that this includes both subparts (i) and (ii) in part (d).

5. (30 points) Consider the Lotka-Volterra predator-prey equations with self-limitation of the prey population:

$$\dot{N} = N(1 - \epsilon N - P)$$
$$\dot{P} = aP(N - 1).$$

Note that the difference between this model and the (non-dimensionalized) one appearing in problem 1 in ODE2 of the notes is the natural *quadratic* death term for the prey equation. Here ϵ and a are *positive* parameters.

(a) Find all steady states of the above system, as well as the parameter regimes for which these steady states are physical, i.e. contain non-negative components.

(b) Show that the coexistence steady state (i.e. the steady state with both components **positive**) is stable, when it exists.

(c) Consider the parameter a as fixed. For what values of $\epsilon = \epsilon(a)$ is the steady state in (b) a spiral sink? For what values is it a stable node? *Hint:* Look in the determinant and trace plane. You want to be above (below) a specific curve...

Bonus (10 possible points) (ES, p.135-136, #5) Problem 5 in the ODE3 section of the notes. Note that this is just a modeling problem, and no calculations are required. However, to receive any points, you must justify your assumptions, i.e. give at least one sentence of explanation for the chosen forms for your equations.